Solutions

1. Changement de variable

$u = 1 + x \Rightarrow x = u - 1$

$du = dx$

$$\int \frac{x}{\sqrt{1+x}}\,dx = \int \frac{u-1}{u^{1/2}}\,du = \int \left[u^{1/2} - u^{-1/2}\right] du$$

$$= \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}\sqrt{(1+x)^3} - 2\sqrt{1+x} + C.$$

Intégration par partie

$u = x$ $\cancel{dx}$ $\quad dv = \frac{1}{\sqrt{1+x}}\,dx$

$du = dx \quad v = 2\sqrt{1+x}$

$$\int \frac{x}{\sqrt{1+x}}\,dx = 2x\sqrt{1+x} - 2\int \sqrt{1+x}\,dx$$

$$= 2x\sqrt{1+x} - \frac{4}{3}\sqrt{(1+x)^3} + C_2$$

On peut montrer algébriquement que les 2 solutions sont équivalentes.

$$\int_0^3 \frac{x}{\sqrt{1+x}}\,dx = 2x\sqrt{1+x} - \frac{4}{3}\sqrt{(1+x)^3}\,\Big|_0^3$$

$$= 6\sqrt{4} - \frac{4}{3}\sqrt{4^3} - \left(0 - \frac{4}{3}\right)$$

$$= 12 - \frac{32}{3} + \frac{4}{3} = \boxed{\frac{8}{3}}$$

2. $\int \sin^3\theta \cos^3\theta \, d\theta$

$= \int \sin^3\theta \cos^2\theta \cos\theta \, d\theta$

$= \int \sin^3\theta (1-\sin^2\theta)\cos\theta \, d\theta$

$u = \sin\theta$

$du = \cos\theta \, d\theta$

$= \int u^3 (1-u^2) \, du$

$= \int [u^3 - u^5] \, du$

$= \frac{u^4}{4} - \frac{u^6}{6} + C_1$

$= \frac{\sin^4\theta}{4} - \frac{\sin^6\theta}{6} + C_1$ (A)

$= \int \cos^3\theta \sin^2\theta \sin\theta \, d\theta$

$= \int \cos^3\theta (1-\cos^2\theta) \sin\theta \, d\theta$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$= -\int u^3 (1-u^2) \, du$

$= \int [u^5 - u^3] \, du$

$= \frac{u^6}{6} - \frac{u^4}{4} + C_2$

$= \frac{\cos^6\theta}{6} - \frac{\cos^4\theta}{4} + C_2$ (B)

Montrons que $A = B \Rightarrow A = \frac{(\sin^2\theta)^2}{4} - \frac{(\sin^2\theta)^3}{6} + C_1$

$= \frac{(1-\cos^2\theta)^2}{4} - \frac{(1-\cos^2\theta)^3}{6} + C_1$

$= \frac{1 - 2\cos^2\theta + \cos^4\theta}{4} - \frac{1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta}{6} + C_1$

$= \frac{3 - \cancel{6\cos^2\theta} + 3\cos^4\theta - 2 + \cancel{6\cos^2\theta} - 6\cos^4\theta + 2\cos^6\theta}{12} + C_1$

$= \frac{3 - 2 - 3\cos^4\theta + 2\cos^6\theta}{12} + C_1$

$= \frac{\cos^6\theta}{6} - \frac{\cos^4\theta}{4} + \overbrace{\frac{1}{12} + C_1}^{C_2}$

$= B$

$\square$

$$\int_{\pi/2}^{\pi} \sin^3\theta \cos^3\theta \, d\theta = \frac{\cos^6\theta}{6} - \frac{\cos^4\theta}{4} \Big|_{\pi/2}^{\pi}$$

$$= \frac{(-1)^6}{6} - \frac{(-1)^4}{4} - (0 - 0)$$

$$= \frac{1}{6} - \frac{1}{4} = \boxed{\frac{-1}{12}}$$

3. $\int \Phi(x)\, dx = \int k x e^x dx = k \int x e^x du$ *

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

donc * $= k \left[x e^x - \int e^x dx \right]$

$= k \left[x e^x - e^x \right] + C$

$$\int_0^1 \Phi(x)\, dx = 1 \Rightarrow k \left[x e^x - e^x \right]_0^1 = 1$$

$$\Rightarrow x e^x - e^x \Big|_0^1 = \frac{1}{k}$$

$$\Leftrightarrow e - e - (0 - 1) = \frac{1}{k}$$

$$\Leftrightarrow 1 = \frac{1}{k} \Leftrightarrow \boxed{k = 1}$$

D'où

$$\int_1^3 x e^x dx = x e^x - e^x \Big|_1^3$$

$$= 3e^3 - e^3 - (e - e)$$

$$= 2e^3$$