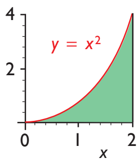
Solutionnaire
Exercices récapitulatifs

Chapitre 5 (page 301)

1.



a) Autour de l'axe des x

Méthode du disque :

$$V = \int_0^2 \pi y^2\,dx = \pi\int_0^2 x^4\,dx = \pi\frac{x^5}{5}\Big|_0^2 = \frac{32\pi}{5}\text{ u}^3$$

Méthode du tube :

$$V = \int_0^4 2\pi y(2-x)\,dy = 2\pi\int_0^4 y\left(2-\sqrt{y}\right)dy = 2\pi\int_0^4\left(2y - y^{\frac{3}{2}}\right)dy = 2\pi\left(y^2 - \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^4 = \frac{32\pi}{5}\text{ u}^3$$

b) Autour de l'axe des y

Méthode du disque :

$$V = 2\pi(2)4 - \int_0^4 \pi x^2\,dy = 16\pi - \pi\int_0^4 y\,dy = 16\pi - \pi\frac{y^2}{2}\Big|_0^4 = 16\pi - 8\pi = 8\pi\text{ u}^3$$

Méthode du tube :

$$V = \int_0^2 2\pi xy\,dx = 2\pi\int_0^2 x^3\,dx = 2\pi\frac{x^4}{4}\Big|_0^2 = 8\pi\text{ u}^3$$

c) Autour de $y = 4$

Méthode du disque :

$$V = \pi(4)^2 2 - \int_0^2 \pi(4-y)^2\,dx = 32\pi - \pi\int_0^2(16 - 8x^2 + x^4)\,dx = 32\pi - \pi\left(16x - \frac{8x^3}{3} + \frac{x^5}{5}\right)\Big|_0^2 = 32\pi - \pi\left(32 - \frac{64}{3} + \frac{32}{5}\right) = \frac{224}{15}\pi\text{ u}^3$$

Méthode du tube :

$$V = \int_0^4 2\pi(4-y)(2-x)\,dy = 2\pi\int_0^4\left(8 - 2y - 4y^{\frac{1}{2}} + y^{\frac{3}{2}}\right)dy = 2\pi\left(8y - y^2 - \frac{8}{3}y^{\frac{3}{2}} + \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^4 = 2\pi\left(32 - 16 - \frac{8}{3}(8) + \frac{2}{5}(32)\right) = \frac{224}{15}\pi\text{ u}^3$$

d) Autour de $y = 5$

Méthode du disque :

$$V = \pi(5)^2 2 - \int_0^2 \pi(5-y)^2\,dx = 50\pi - \pi\int_0^2(25 - 10x^2 + x^4)\,dx = 50\pi - \pi\left(25x - \frac{10x^3}{3} + \frac{x^5}{5}\right)\Big|_0^2 = 50\pi - \pi\left(50 - \frac{10}{3}(8) + \frac{32}{5}\right) = \frac{304\pi}{15}\text{ u}^3$$

Méthode du tube :

$$V = \int_0^4 2\pi(5-y)(2-x)\,dy = 2\pi\int_0^4\left(10 - 2y - 5y^{\frac{1}{2}} + y^{\frac{3}{2}}\right)dy = 2\pi\left(10y - y^2 - \frac{10}{3}y^{\frac{3}{2}} + \frac{2}{5}y^{\frac{5}{2}}\right)\Big|_0^4 = 2\pi\left(40 - 16 - \frac{10}{3}(8) + \frac{2}{5}(32)\right) = \frac{304\pi}{15}\text{ u}^3$$

e) Autour de $x = 2$

Méthode du disque :

$$V = \int_0^4 \pi(2-x)^2\,dy = \pi\int_0^4\left(4 - 4y^{\frac{1}{2}} + y\right)dy = \pi\left(4y - \frac{8}{3}y^{\frac{3}{2}} + \frac{y^2}{2}\right)\Big|_0^4 = \pi\left(16 - \frac{8}{3}(8) + 8\right) = \frac{8\pi}{3}\text{ u}^3$$

Méthode du tube :

$$V = \int_0^2 2\pi(2-x)y\,dx$$
$$= 2\pi\int_0^4 (2x^2 - x^3)\,dx$$
$$= 2\pi\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\Bigg|_0^4$$
$$= 2\pi\left(\frac{2}{3}(64) - 64\right) = \frac{8\pi}{3}\text{ u}^3$$

f) Autour de $x = -2$

Méthode du disque :

$$V = \pi(4)^2 4 - \int_0^4 \pi\left(x - (-2)\right)^2 dy$$
$$= 64\pi - \pi\int_0^4\left(4 + 4y^{\frac{1}{2}} + y\right)dy$$
$$= 64\pi - \pi\left(4y + \frac{8}{3}y^{\frac{3}{2}} + \frac{y^2}{2}\right)\Bigg|_0^4$$
$$= 64\pi - \pi\left(16 + \frac{8}{3}(8) + 8\right) = \frac{56\pi}{3}\text{ u}^3$$

Méthode du tube :

$$V = \int_0^2 2\pi\left(x - (-2)\right)y\,dx$$
$$=2\pi\int_0^2 (x^3 + 2x^2)\,dx$$
$$= 2\pi\left(\frac{x^4}{4} + \frac{2x^3}{3}\right)\Bigg|_0^2$$
$$= 2\pi\left(4 + \frac{2}{3}(8)\right) = \frac{56\pi}{3}\text{ u}^3$$

g) Autour de $y = -2$

Méthode du disque :

$$V = \int_0^2 \pi\left(y - (-2)\right)^2 dx - \pi(2)^2 2$$
$$= \pi\int_0^2 (x^4 + 4x^2 + 4)\,dx - 8\pi$$
$$= \pi\left(\frac{x^5}{5} + \frac{4x^3}{3} + 4x\right)\Bigg|_0^2 - 8\pi$$
$$= \pi\left(\frac{32}{5} + \frac{4}{3}(8) + 8\right) - 8\pi$$
$$= \frac{256\pi}{15}\text{ u}^3$$

Méthode du tube :

$$V = \int_0^4 2\pi\left(y - (-2)\right)(2 - x)\,dy$$
$$= 2\pi\int_0^4\left(2y - y^{\frac{3}{2}} + 4 - 2y^{\frac{1}{2}}\right)dy$$
$$= 2\pi\left(y^2 - \frac{2}{5}y^{\frac{5}{2}} + 4y - \frac{4}{3}y^{\frac{3}{2}}\right)\Bigg|_0^4$$
$$= 2\pi\left(16 - \frac{2}{5}(32) + 16 - \frac{4}{3}(8)\right)$$
$$= \frac{256\pi}{15}\text{ u}^3$$

h) Autour de $x = 6$

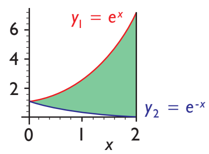
Méthode du disque :

$$V = \int_0^4 \pi(6-x)^2\,dy - \pi(4)^2 4$$
$$= \pi\int_0^4\left(36 - 12y^{\frac{1}{2}} + y\right)dy - 64\pi$$
$$= \pi\left(36y - 8y^{\frac{3}{2}} + \frac{y^2}{2}\right)\Bigg|_0^4 - 64\pi$$
$$= \pi\,(144 - 8(8) + 8) - 64\pi$$
$$= 24\pi\text{ u}^3$$

Méthode du tube :

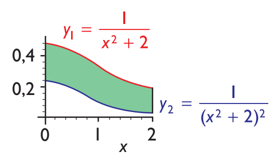
$$V = \int_0^2 2\pi(6-x)y\,dx$$
$$= 2\pi\int_0^2 (6x^2 - x^3)\,dx$$
$$= 2\pi\left(2x^3 - \frac{x^4}{4}\right)\Bigg|_0^2 = 24\pi\text{ u}^3$$

2. a)



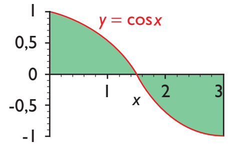
$$V = V_1 - V_2$$
$$= \pi\int_0^2 (e^x)^2\,dx - \pi\int_0^2 (e^{-x})^2\,dx$$
$$= \pi\int_0^2 e^{2x}\,dx - \pi\int_0^2 e^{-2x}\,dx$$
$$= \pi\left(\frac{e^{2x}}{2}\Bigg|_0^2\right) - \pi\left(\frac{-e^{-2x}}{2}\Bigg|_0^2\right)$$
$$= \pi\left(\frac{e^4 - 1}{2}\right) - \pi\left(\frac{-e^{-4} + 1}{2}\right)$$
$$= \pi\left(\frac{e^4 + e^{-4}}{2} - 1\right) \approx 82{,}6\text{ u}^3$$

b)

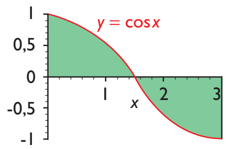


$$V = 2\pi\int_0^2 x(y_1 - y_2)\,dx$$
$$= 2\pi\int_0^2 x\left(\frac{1}{x^2+2} - \frac{1}{(x^2+2)^2}\right)dx$$
$$= 2\pi\int_0^2\left(\frac{x}{x^2+2} - \frac{x}{(x^2+2)^2}\right)dx$$
$$= 2\pi\left(\frac{1}{2}\ln\left|x^2+2\right| + \frac{1}{2(x^2+2)}\right)\Bigg|_0^2$$
$$= 2\pi\left[\left(\frac{1}{2}\ln 6 + \frac{1}{12}\right) - \left(\frac{1}{2}\ln 2 + \frac{1}{4}\right)\right]$$
$$= \pi\left(\ln 3 - \frac{1}{3}\right) \approx 2{,}4\text{ u}^3$$

c) $$V = 2\left\{\pi\int_0^{\frac{\pi}{2}} y^2\,dx\right\}$$
$$= 2\pi\int_0^{\frac{\pi}{2}} (\cos x)^2\,dx$$
$$= 2\pi\int_0^{\frac{\pi}{2}} \frac{1+\cos 2x}{2}\,dx$$
$$= 2\pi\left(\frac{x}{2}+\frac{\sin 2x}{4}\right)\Bigg|_0^{\frac{\pi}{2}}$$
$$= 2\pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2}\ \text{u}^3$$

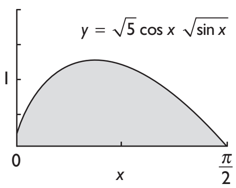


d)



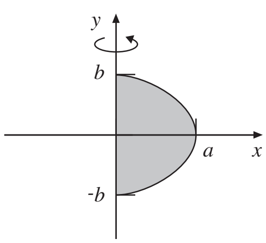
$$V = 2\pi\int_0^{\frac{\pi}{2}} x(y-0)\,dx + 2\pi\int_{\frac{\pi}{2}}^{\pi} x(0-y)\,dx$$
$$= 2\pi\int_0^{\frac{\pi}{2}} x\cos x\,dx - 2\pi\int_{\frac{\pi}{2}}^{\pi} x\cos x\,dx$$
$$= 2\pi(x\sin x+\cos x)\Big|_0^{\frac{\pi}{2}} - 2\pi(x\sin x+\cos x)\Big|_{\frac{\pi}{2}}^{\pi}$$
$$= 2\pi\left(\frac{\pi}{2}-1\right) - 2\pi\left(-1-\frac{\pi}{2}\right) = 2\pi^2\ \text{u}^3$$

e)



$$V = \pi\int_0^{\frac{\pi}{2}} \left(\sqrt{5}\cos x\sqrt{\sin x}\right)^2 dx$$
$$= \pi\int_0^{\frac{\pi}{2}} 5\cos^2 x\sin x\,dx$$
$$= 5\pi\left(\frac{-\cos^3 x}{3}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{5\pi}{3}\ \text{u}^2$$

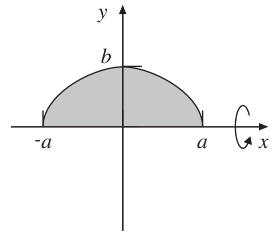
3. a)



Puisque $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $x^2 = a^2\left(1-\dfrac{y^2}{b^2}\right)$

$$V = \int_{-b}^{b} \pi x^2\,dy$$
$$= \pi a^2\int_{-b}^{b}\left(1-\frac{y^2}{b^2}\right)dy$$
$$= \pi a^2\left(y-\frac{y^3}{3b^2}\right)\Bigg|_{-b}^{b}$$
$$= \frac{4\pi a^2 b}{3}\ \text{u}^3$$

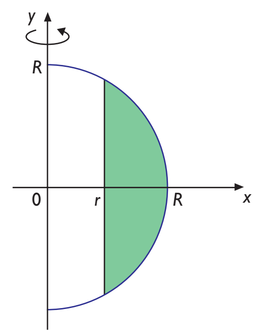
b)



Puisque $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, $y^2 = b^2\left(1-\dfrac{x^2}{a^2}\right)$

$$V = \int_{-a}^{a} \pi y^2\,dx$$
$$= \pi b^2\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)dx$$
$$= \pi b^2\left(x-\frac{x^3}{3a^2}\right)\Bigg|_{-a}^{a}$$
$$= \frac{4\pi ab^2}{3}\ \text{u}^3$$

4. a)



$$V = \int_r^R 2\pi x(2y)\,dx$$
$$= 4\pi\int_r^R x\sqrt{R^2-x^2}\,dx$$
$$= -4\pi\left(\frac{1}{3}\right)(R^2-x^2)^{\frac{3}{2}}\Bigg|_r^R$$
$$= \frac{-4\pi}{3}\left[0-(R^2-r^2)^{\frac{3}{2}}\right]$$
$$= \frac{4\pi}{3}\sqrt{(R^2-r^2)^3}\ \text{u}^3$$

b) Si $r = \dfrac{R}{2}$, le volume V_1 restant est

$$V_1 = \frac{4\pi}{3}\sqrt{\left(R^2-\frac{R^2}{4}\right)^3} = \frac{\pi\sqrt{3}}{2}R^3$$

Le volume V_2 enlevé est

$$V_2 = V_{\text{sphère}} - V_1$$
$$= \frac{4}{3}\pi R^3 - \frac{\pi\sqrt{3}}{2}R^3$$

d'où $V_2 = \pi\left(\dfrac{8-3\sqrt{3}}{6}\right)R^3\ \text{u}^3$

c) Si $R = 2$, alors

$$V_{\text{total}} = \frac{4\pi}{3}(2)^3 = \frac{32\pi}{3}\ \text{cm}^3$$

$$\frac{1}{2}V_{\text{total}} = \frac{16\pi}{3}\ \text{cm}^3$$

$$\text{Ainsi } \frac{4\pi}{3}(2^2 - r^2)^{\frac{3}{2}} = \frac{16\pi}{3} \quad \text{(voir a))}$$

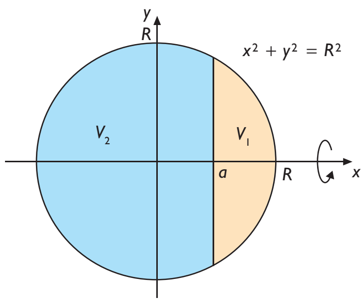
$$(4 - r^2)^{\frac{3}{2}} = 4$$

$$4 - r^2 = 4^{\frac{2}{3}}$$

$$r^2 = 4 - 4^{\frac{2}{3}}$$

$$r \approx 1{,}2 \text{ cm}$$

5. a) Soit le cercle d'équation $x^2 + y^2 = R^2$ que nous faisons tourner autour de l'axe des x.



$$V_1 = \pi\int_a^R y^2\,dx$$

$$= \pi\int_a^R (R^2 - x^2)\,dx$$

$$= \pi\left(R^2x - \frac{x^3}{3}\right)\Bigg|_a^R$$

$$= \pi\left[\left(R^3 - \frac{R^3}{3}\right) - \left(aR^2 - \frac{a^3}{3}\right)\right]$$

$$= \pi\left(\frac{2R^3}{3} - aR^2 + \frac{a^3}{3}\right)\text{u}^3$$

$$V_2 = \text{Volume de la sphère} - V_1$$

$$= \frac{4\pi R^3}{3} - \pi\left(\frac{2R^3}{3} - aR^2 + \frac{a^3}{3}\right)$$

$$= \pi\left(\frac{2R^3}{3} + aR^2 - \frac{a^3}{3}\right)\text{u}^3$$

b) Soit $R = 10$ mètres.

Si la sphère contient 2 mètres d'eau de hauteur, le volume V_3 obtenu correspond à V_1 avec $a = 8$.

$$\text{Ainsi } V_3 = \pi\left(\frac{2(10)^3}{3} - 8(10)^2 + \frac{(8)^3}{3}\right)$$

$$\text{d'où } V_3 = \frac{112\pi}{3}\ \text{m}^3$$

Si la sphère contient 13 mètres d'eau de hauteur, le volume V_4 obtenu correspond à V_2 avec $a = 3$.

$$\text{Ainsi } V_4 = \pi\left(\frac{2(10)^3}{3} + 3(10)^2 - \frac{(3)^3}{3}\right)$$

$$\text{d'où } V_4 = \frac{2873\pi}{3}\ \text{m}^3$$

c) $$m_1 = \left(\frac{112\pi}{3}\ \text{m}^3\right)(1000\ \text{kg/m}^3) \approx 117\,286{,}13\ \text{kg}$$

$$m_2 = \left(\frac{2873\pi}{3}\ \text{m}^3\right)(1000\ \text{kg/m}^3) \approx 3\,008\,598{,}6\ \text{kg}$$

d) Calculons d'abord le volume V_5 contenu entre $a = \frac{R}{2}$ et R.

En remplaçant a par $\frac{R}{2}$ dans V_1, nous obtenons

$$V_5 = \pi\left(\frac{2R^3}{3} - \left(\frac{R}{2}\right)R^2 + \frac{\left(\frac{R}{2}\right)^3}{3}\right)$$

$$\text{Ainsi } V_5 = \frac{5\pi R^3}{24}$$

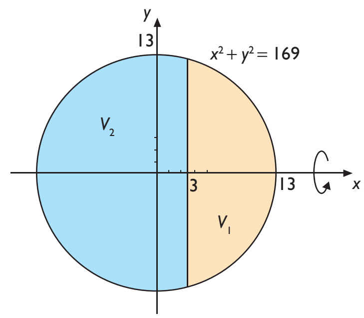
Le volume $V_{\frac{1}{2}}$ de la demi-sphère est $\frac{2\pi R^3}{3}$

donc le pourcentage d'espace occupé est $\frac{V_5}{V_{\frac{1}{2}}}$, où

$$\frac{V_5}{V_{\frac{1}{2}}} = \frac{\frac{5\pi R^3}{24}}{\frac{2\pi R^3}{3}} = 0{,}3125$$

d'où 31,25 %

e) En utilisant le calcul intégral



$$V_1 = \pi\int_3^{13} y^2\,dx$$

$$= \pi\int_3^{13}(169 - x^2)\,dx$$

$$= \pi\left(169x - \frac{x^3}{3}\right)\Bigg|_3^{13} = \frac{2900}{3}\pi\ \text{u}^3$$

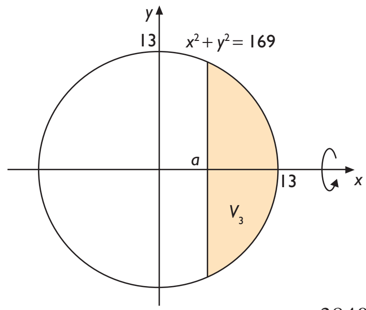
$$V_2 = \text{Volume sphère (rayon 13)} - V_1$$

$$= \frac{4}{3}\pi 13^3 - \frac{2900}{3}\pi = \frac{5888\pi}{3}\ \text{u}^3$$

$$V_3 = V_2 - \text{Volume sphère (rayon 8)}$$

$$= \frac{5888\pi}{3} - \frac{4\pi(8)^3}{3} = \frac{3840\pi}{3}\ \text{u}^3$$

5



$$V_3 = \frac{3840\pi}{3}$$

$$\pi\int_a^{13} y^2\,dx = \frac{3840\pi}{3}$$

$$\pi\int_a^{13} (169 - x^2)\,dx = \frac{3840\pi}{3}$$

$$\pi\left(169x - \frac{x^3}{3}\right)\Bigg|_a^{13} = \frac{3840\pi}{3}$$

$$(169)13 - \frac{13^3}{3} - 169a + \frac{a^3}{3} = \frac{3840}{3}$$

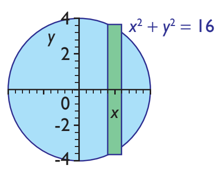
$$a^3 - 507a + 554 = 0$$

$a \approx 1{,}095$ (outil technologique)

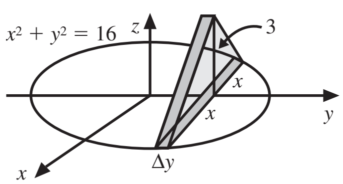
Hauteur d'eau $= 13 - a$

d'où environ 11,905 cm

6. a) Base du solide :



Section du solide :



Volume de la section $= \begin{pmatrix}\text{aire du}\\ \text{triangle}\end{pmatrix} \cdot \begin{pmatrix}\text{épaisseur}\\ \text{de la base}\end{pmatrix}$

$$\Delta V = \frac{(2x)(3)}{2}\Delta y$$

Ainsi $V = \int_{-4}^{4} 3x\,dy$

$= 3\int_{-4}^{4}\sqrt{16 - y^2}\,dy$ (car $x^2 + y^2 = 16$)

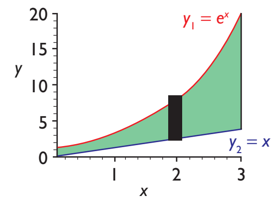
$= 3\left(\frac{y}{2}\sqrt{16 - y^2} + \frac{16}{2}\,\text{Arc sin}\left(\frac{y}{4}\right)\right)\Bigg|_{-4}^{4}$

$= 3(8\,\text{Arc sin}(1) - 8\,\text{Arc sin}(-1))$

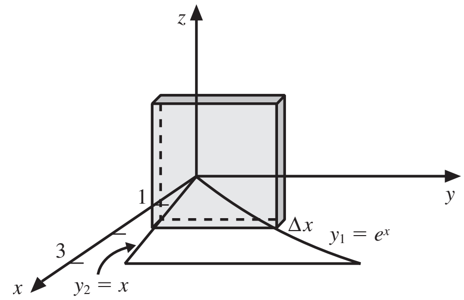
$= 3\left(8\left(\frac{\pi}{2}\right) - 8\left(\frac{-\pi}{2}\right)\right)$

$= 24\pi$ u^3

b) Base du solide :



Section du solide :



Volume de la section $= \begin{pmatrix}\text{aire du}\\ \text{carré}\end{pmatrix} \cdot \begin{pmatrix}\text{épaisseur}\\ \text{de la base}\end{pmatrix}$

$= (y_1 - y_2)^2 \cdot \Delta x$

$= (e^x - x)^2\,\Delta x$

Ainsi $V = \int_0^3 (e^{2x} - 2xe^x + x^2)\,dx$

$= \int_0^3 e^{2x}\,dx - 2\int_0^3 xe^x\,dx + \int_0^3 x^2\,dx$

$= \frac{e^{2x}}{2}\Bigg|_0^3 - 2(xe^x - e^x)\Big|_0^3 + \frac{x^3}{3}\Bigg|_0^3$

$= \left(\frac{e^6}{2} - \frac{1}{2}\right) - 2(2e^3 + 1) + 9$

$= \left(\frac{e^6}{2} - 4e^3 + \frac{13}{2}\right)$

$\approx 127{,}9$ u^3

7. a) $L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_0^1 \sqrt{1 + (2x)^2}\,dx$ $\left(\text{car } \frac{dy}{dx} = 2x\right)$

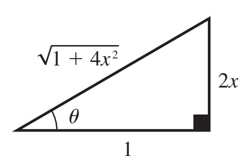
$= \int_0^1 \sqrt{1 + 4x^2}\,dx$

$4x^2 = \tan^2\theta$

$2x = \tan\theta$

$dx = \frac{\sec^2\theta}{2}\,d\theta$

$\theta = \text{Arc tan}\,(2x)$



$\int\sqrt{1 + 4x^2}\,dx = \int\sqrt{1 + \tan^2\theta}\,\frac{\sec^2\theta}{2}\,d\theta$

$= \frac{1}{2}\int\sec^3\theta\,d\theta$

$= \frac{1}{2}\left[\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}\right] + C$

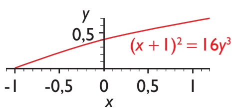
$= \frac{1}{4}\left[2x\sqrt{1 + 4x^2} + \ln\left|2x + \sqrt{1 + 4x^2}\right|\right] + C$

$$\text{Ainsi } L = \frac{1}{4}\left(2x\sqrt{1+4x^2} + \ln\left|2x+\sqrt{1+4x^2}\right|\right)\Big|_0^1$$
$$= \frac{1}{4}\left(2\sqrt{5} + \ln\left(2+\sqrt{5}\right)\right)$$
$$= \frac{2\sqrt{5} + \ln\left(2+\sqrt{5}\right)}{4}$$
$$\approx 1{,}48 \text{ u}$$

$$L \text{ segment} = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2} \approx 1{,}41 \text{ u}$$

b) Exprimons x en fonction de y.

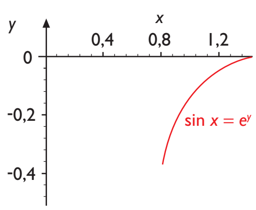
$$(x+1) = \pm 4y^{\frac{3}{2}}$$
$$x = -1 \pm 4y^{\frac{3}{2}}$$
$$\text{Ainsi } x = -1 + 4y^{\frac{3}{2}} \qquad (\text{car } x \geq -1)$$



$$L = \int_0^{\frac{2}{3}} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$
$$= \int_0^{\frac{2}{3}} \sqrt{1 + \left(6y^{\frac{1}{2}}\right)^2}\, dy$$
$$= \int_0^{\frac{2}{3}} \sqrt{1 + 36y}\, dy$$
$$= \frac{(1+36y)^{\frac{3}{2}}}{54}\Big|_0^{\frac{2}{3}}$$
$$= \frac{(25)^{\frac{3}{2}}}{54} - \frac{(1)^{\frac{3}{2}}}{54}$$
$$= \frac{62}{27} \text{ u}$$

c) Exprimons y en fonction de x.

$$e^y = \sin x$$
$$\text{Ainsi } y = \ln(\sin x)$$



$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 + \left(\frac{\cos x}{\sin x}\right)^2}\, dx$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{1 + \cot^2 x}\, dx$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc x\, dx$$
$$= -\ln\left|\csc x + \cot x\right|\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$
$$= -\ln|1| + \ln\left|\sqrt{2} + 1\right|$$
$$= \ln\left|\sqrt{2} + 1\right|$$
$$\approx 0{,}88 \text{ u}$$

d) $$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{\left[\frac{d}{dt}(e^t \sin t)\right]^2 + \left[\frac{d}{dt}(e^t \cos t)\right]^2}\, dt$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{(e^t \sin t + e^t \cos t)^2 + (e^t \cos t - e^t \sin t)^2}\, dt$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{2e^{2t}}\, dt$$
$$= \sqrt{2}\int_0^{\frac{\pi}{2}} e^t\, dt$$
$$= \sqrt{2}e^t\Big|_0^{\frac{\pi}{2}}$$
$$= \sqrt{2}\left(e^{\frac{\pi}{2}} - 1\right)$$
$$\approx 5{,}39 \text{ u}$$

e) $$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^1 \sqrt{\left[\frac{d}{dt}(1-2t^2)\right]^2 + \left[\frac{d}{dt}(1+t^3)\right]^2}\, dt$$
$$= \int_0^1 \sqrt{(-4t)^2 + (3t^2)^2}\, dt$$
$$= \int_0^1 \sqrt{16t^2 + 9t^4}\, dt$$
$$= \int_0^1 t\sqrt{16 + 9t^2}\, dt$$
$$= \frac{(16+9t^2)^{\frac{3}{2}}}{27}\Big|_0^1$$
$$= \frac{125}{27} - \frac{64}{27}$$
$$= \frac{61}{27} \text{ u}$$

8. a) $H(x) = 500\left(e^{\frac{x}{1000}} + e^{\frac{-x}{1000}}\right) - 980$

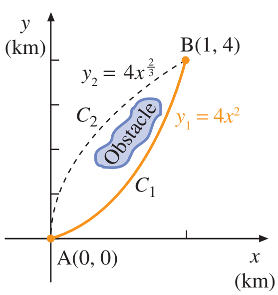
$$H'(x) = \frac{1}{2}\left(e^{\frac{x}{1000}} - e^{\frac{-x}{1000}}\right)$$
$$H'(x) = 0 \text{ si } x = 0$$

x	-100		0		100
$H'(x)$	∄	–	0	+	∄
H		↘	20	↗	
			min.		

d'où la hauteur minimale $H_1 = H(0) = 20$ m;
la hauteur des pylônes $H_2 = H(100) = H(-100) \approx 25$ m

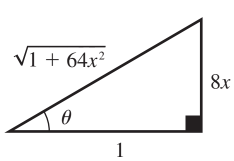
b) $$\begin{aligned}L &= \int_{-100}^{100}\sqrt{1+\left(H'(x)\right)^2}\,dx\\ &= \int_{-100}^{100}\sqrt{1+\left[\frac{1}{2}\left(e^{\frac{x}{1000}}-e^{\frac{-x}{1000}}\right)\right]^2}\,dx\\ &= \int_{-100}^{100}\sqrt{1+\frac{1}{4}\left(e^{\frac{2x}{1000}}-2+e^{\frac{-2x}{1000}}\right)}\,dx\\ &= \int_{-100}^{100}\sqrt{\frac{e^{\frac{2x}{1000}}}{4}+\frac{2}{4}+\frac{e^{\frac{-2x}{1000}}}{4}}\,dx\\ &= \int_{-100}^{100}\sqrt{\frac{1}{4}\left(e^{\frac{x}{1000}}+e^{\frac{-x}{1000}}\right)^2}\,dx\\ &= \int_{-100}^{100}\frac{1}{2}\left(e^{\frac{x}{1000}}+e^{\frac{-x}{1000}}\right)dx\\ &= 500\left(e^{\frac{x}{1000}}-e^{\frac{-x}{1000}}\right)\Big|_{-100}^{100}\\ &= 500\left[\left(e^{\frac{1}{10}}-e^{\frac{-1}{10}}\right)-\left(e^{\frac{-1}{10}}-e^{\frac{1}{10}}\right)\right]\\ &= 1000\left(e^{\frac{1}{10}}-e^{\frac{-1}{10}}\right)\\ &\approx 200{,}3\text{ m}\end{aligned}$$

9. Calculons d'abord la longueur de la courbe C_1 et le coût de construction de C_1.



$$\begin{aligned}L_{C_1} &= \int_0^1\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ &= \int_0^1\sqrt{1+(8x)^2}\,dx\\ &= \int_0^1\sqrt{1+64x^2}\,dx\end{aligned}$$

$$\begin{aligned}64x^2 &= \tan^2\theta\\ 8x &= \tan\theta\\ dx &= \frac{\sec^2\theta}{8}\,d\theta\\ \theta &= \text{Arc tan}\,(8x)\end{aligned}$$



$$\begin{aligned}\int\sqrt{1+64x^2}\,dx &= \int\sqrt{1+\tan^2\theta}\;\frac{\sec^2\theta}{8}\,d\theta\\ &= \frac{1}{8}\int\sec^3\theta\,d\theta\\ &= \frac{1}{8}\left[\frac{\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|}{2}\right]+C\\ &= \frac{1}{16}\left[8x\sqrt{1+64x^2}+\ln\left|8x+\sqrt{1+64x^2}\right|\right]+C\end{aligned}$$

Ainsi $$\begin{aligned}L_{C_1} &= \frac{1}{16}\left(8x\sqrt{1+64x^2}+\ln\left|8x+\sqrt{1+64x^2}\right|\right)\Big|_0^1\\ &= \frac{1}{16}\left(8\sqrt{65}+\ln\left(8+\sqrt{65}\right)\right)\text{ km}\end{aligned}$$

$$\begin{aligned}\text{Coût de } C_1 &= \left(\frac{8\sqrt{65}+\ln\left(8+\sqrt{65}\right)}{16}\right)1\,000\,000\\ &\approx 4\,204\,658{,}40\ \$\end{aligned}$$

Calculons maintenant la longueur de la courbe C_2 et le coût de construction de C_2.

De $y = 4x^{\frac{2}{3}}$ nous obtenons $x = \frac{1}{8}y^{\frac{3}{2}}$.

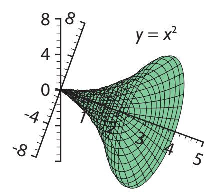
$$\begin{aligned}L_{C_2} &= \int_0^4\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy\\ &= \int_0^4\sqrt{1+\left(\frac{3}{16}y^{\frac{1}{2}}\right)^2}\,dy\\ &= \int_0^4\sqrt{1+\frac{9y}{256}}\,dy\\ &= \frac{1}{16}\int_0^4\sqrt{256+9y}\,dy\\ &= \frac{1}{216}(256+9y)^{\frac{3}{2}}\Big|_0^4\\ &= \frac{1}{216}\left(292^{\frac{3}{2}}-4096\right)\end{aligned}$$

$$\begin{aligned}\text{Coût de } C_2 &= \frac{1}{216}\left(292^{\frac{3}{2}}-4096\right)1\,000\,000\\ &\approx 4\,137\,491{,}60\ \$\end{aligned}$$

Le chemin le plus économique est C_2 et l'économie réalisée E est donnée par

$E \approx 4\,204\,658{,}40 - 4\,137\,491{,}60 \approx 67\,167\ \$$

10. a) i)

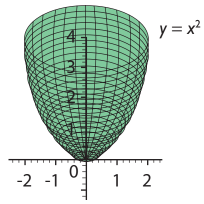


$$\begin{aligned}S_{0X} &= \int_0^3 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\\ &= 2\pi\int_0^3 x^2\sqrt{1+(2x)^2}\,dx\\ &= 2\pi\int_0^3 x^2\sqrt{1+4x^2}\,dx\\ &= 4\pi\int_0^3 x^2\sqrt{\frac{1}{4}+x^2}\,dx\\ &= 4\pi\left(\frac{x}{8}\left(2x^2+\frac{1}{4}\right)\sqrt{x^2+\frac{1}{4}}-\frac{\left(\frac{1}{4}\right)^2}{8}\ln\left|x+\sqrt{x^2+\frac{1}{4}}\right|\right)\Bigg|_0^3\end{aligned}$$

(formule 14, page 473)

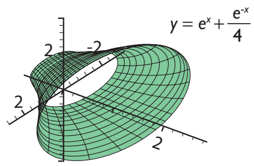
$$\begin{aligned}&= 4\pi\left[\left(\frac{3}{8}(18{,}25)\sqrt{9{,}25}-\frac{1}{128}\ln\left(3+\sqrt{9{,}25}\right)\right)-\left(\frac{1}{128}\ln\sqrt{0{,}25}\right)\right]\\ &\approx 261{,}3\text{ u}^2\end{aligned}$$

ii)



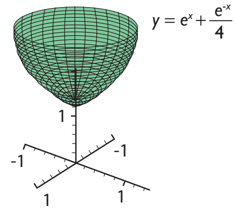
$$S_{0Y} = \int_0^3 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi\int_0^3 x\sqrt{1+4x^2}\,dx$$
$$= \frac{\pi}{6}(1+4x^2)^{\frac{3}{2}}\Big|_0^3$$
$$= \frac{\pi}{6}\left[37^{\frac{3}{2}}-1\right]$$
$$\approx 117{,}3\ \text{u}^2$$

b) i)



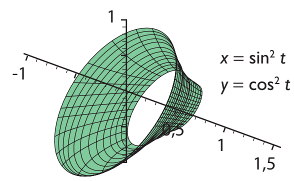
$$S_{0X} = \int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi\int_0^1\left(e^x+\frac{e^{-x}}{4}\right)\sqrt{1+\left(e^x-\frac{e^{-x}}{4}\right)^2}\,dx$$
$$= 2\pi\int_0^1\left(e^x+\frac{e^{-x}}{4}\right)\sqrt{1+e^{2x}-\frac{1}{2}+\frac{e^{-2x}}{16}}\,dx$$
$$= 2\pi\int_0^1\left(e^x+\frac{e^{-x}}{4}\right)\sqrt{e^{2x}+\frac{1}{2}+\frac{e^{-2x}}{16}}\,dx$$
$$= 2\pi\int_0^1\left(e^x+\frac{e^{-x}}{4}\right)\sqrt{\left(e^x+\frac{e^{-x}}{4}\right)^2}\,dx$$
$$= 2\pi\int_0^1\left(e^x+\frac{e^{-x}}{4}\right)\left(e^x+\frac{e^{-x}}{4}\right)dx$$
$$= 2\pi\int_0^1\left(e^{2x}+\frac{1}{2}+\frac{e^{-2x}}{16}\right)dx$$
$$= 2\pi\left(\frac{e^{2x}}{2}+\frac{x}{2}-\frac{e^{-2x}}{32}\right)\Big|_0^1$$
$$= 2\pi\left(\frac{e^2}{2}-\frac{e^{-2}}{32}+\frac{1}{32}\right)$$
$$\approx 23{,}4\ \text{u}^2$$

ii)



$$S_{0Y} = \int_0^1 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi\int_0^1 x\sqrt{1+\left(e^x-\frac{e^{-x}}{4}\right)^2}\,dx$$
$$= 2\pi\int_0^1 x\left(e^x+\frac{e^{-x}}{4}\right)dx \quad (\text{voir } S_{0X})$$
$$= 2\pi\left[\int_0^1 xe^x\,dx-\frac{1}{4}\int_0^1 xe^{-x}\,dx\right]$$
$$= 2\pi\left[(xe^x-e^x)\Big|_0^1-\frac{1}{4}(xe^{-x}+e^{-x})\Big|_0^1\right]$$
$$= 2\pi\left[1-\frac{1}{4}(2e^{-1}-1)\right]$$
$$= 2\pi\left(\frac{5}{4}-\frac{1}{2e}\right)$$
$$\approx 6{,}7\ \text{u}^2$$

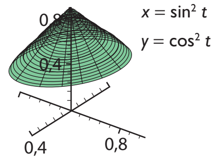
c) i)



$$S_{0X} = \int_0^{\frac{\pi}{4}} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$= 2\pi\int_0^{\frac{\pi}{4}}\cos^2 t\sqrt{(2\sin t\cos t)^2+(-2\sin t\cos t)^2}\,dt$$
$$= 2\pi\int_0^{\frac{\pi}{4}}\cos^2 t\sqrt{8\sin^2 t\cos^2 t}\,dt$$
$$= 4\sqrt{2}\pi\int_0^{\frac{\pi}{4}}\cos^3 t\sin t\,dt$$
$$= 4\sqrt{2}\pi\left(\frac{-\cos^4 t}{4}\Big|_0^{\frac{\pi}{4}}\right)$$
$$= 4\sqrt{2}\pi\left(\frac{-1}{16}+\frac{1}{4}\right)$$
$$\approx 3{,}3\ \text{u}^2$$

5

ii)



$$\begin{aligned} S_{0Y} &= \int_0^{\frac{\pi}{4}} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= 2\pi\int_0^{\frac{\pi}{4}} \sin^2 t\sqrt{8\sin^2 t\cos^2 t}\; dt \\ &= 4\sqrt{2}\pi\int_0^{\frac{\pi}{4}} \sin^3 t\cos t\, dt \\ &= 4\sqrt{2}\pi\left(\frac{\sin^4 t}{4}\Bigg|_0^{\frac{\pi}{4}}\right) \\ &= 4\sqrt{2}\pi\left(\frac{1}{16}\right) \\ &\approx 1{,}1\text{ u}^2 \end{aligned}$$

11. Soit le cercle $x^2 + y^2 = 625$, le cercle de rayon 5 que l'on fait tourner autour de l'axe des y, où $y \in [10, 25]$ pour engendrer la calotte.

$$\begin{aligned} S_{0Y} &= \int_0^{\sqrt{525}} 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \left(x \in \left[0, \sqrt{525}\right]\right) \\ &= 2\pi\int_0^{\sqrt{525}} x\sqrt{1 + \left(\frac{-x}{\sqrt{625 - x^2}}\right)^2}\, dx \quad \left(y = \sqrt{625 - x^2}\right) \\ &= 2\pi\int_0^{\sqrt{525}} x\sqrt{1 + \frac{x^2}{625 - x^2}}\, dx \\ &= 2\pi\int_0^{\sqrt{525}} x\sqrt{\frac{625}{625 - x^2}}\, dx \\ &= 50\pi\int_0^{\sqrt{525}} \frac{x}{\sqrt{625 - x^2}}\, dx \\ &= 50\pi\left(-\sqrt{625 - x^2}\,\Big|_0^{\sqrt{525}}\right) \\ &= 750\pi \end{aligned}$$

Coût d'achat $= (750\pi \text{ m}^2)\left(\dfrac{1}{10\text{ m}^2}\right)16\,\$$

d'où environ 3769,91 $

12. a)

$$\begin{aligned} \int_0^1 \frac{\left(1 + \sqrt{x}\right)^5}{\sqrt{x}}\, dx &= \lim_{s\to 0^+}\int_s^1 \frac{\left(1 + \sqrt{x}\right)^5}{\sqrt{x}}\, dx \qquad \left(u = 1 + \sqrt{x}\right) \\ &= \lim_{s\to 0^+}\left(\frac{\left(1 + \sqrt{x}\right)^6}{3}\Bigg|_s^1\right) \\ &= \lim_{s\to 0^+}\left(\frac{2^6}{3} - \frac{\left(1 + \sqrt{s}\right)^6}{3}\right) \\ &= \left(\frac{2^6}{3} - \frac{1}{3}\right) \\ &= 21 \end{aligned}$$

L'intégrale est convergente.

b)

$$\begin{aligned} \int_{-\infty}^0 \frac{x^2}{x^2 + 1}\, dx &= \lim_{N\to -\infty}\int_N^0 \left(1 - \frac{1}{x^2 + 1}\right) dx \\ &= \lim_{N\to -\infty}(x - \text{Arc tan } x)\Big|_N^0 \\ &= \lim_{N\to -\infty}\left[(0 - \text{Arc tan } 0) - (N - \text{Arc tan } N)\right] \\ &= \left(0 - \left(-\infty + \frac{\pi}{4}\right)\right) \\ &= +\infty \end{aligned}$$

L'intégrale est divergente.

c)

$$\begin{aligned} \int_1^{+\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{x^2}\, dx &= \lim_{M\to +\infty}\int_1^M \frac{\sin\left(\frac{\pi}{x}\right)}{x^2}\, dx \qquad \left(u = \frac{\pi}{x}\right) \\ &= \lim_{M\to +\infty}\left(\frac{\cos\left(\frac{\pi}{x}\right)}{\pi}\Bigg|_1^M\right) \\ &= \lim_{M\to +\infty}\left(\frac{\cos\left(\frac{\pi}{M}\right)}{\pi} - \frac{\cos(\pi)}{\pi}\right) \\ &= \frac{1}{\pi} - \left(\frac{-1}{\pi}\right) \\ &= \frac{2}{\pi} \end{aligned}$$

L'intégrale est convergente.

d)

$$\begin{aligned} \int_{-1}^8 \frac{1}{\sqrt[3]{x^5}}\, dx &= \int_{-1}^0 x^{\frac{-5}{3}}\, dx + \int_0^8 x^{\frac{-5}{3}}\, dx \\ &= \lim_{t\to 0^-}\int_{-1}^t x^{\frac{-5}{3}}\, dx + \lim_{s\to 0^+}\int_s^8 x^{\frac{-5}{3}}\, dx \\ &= \lim_{t\to 0^+}\left(\frac{-3}{2}t^{\frac{-2}{3}}\Big|_{-1}^t\right) + \lim_{s\to 0^+}\left(\frac{-3}{2}s^{\frac{-2}{3}}\Big|_s^8\right) \\ &= \lim_{t\to 0^+}\left(\frac{-3}{2\sqrt[3]{t^2}} + \frac{3}{2}\right) + \lim_{s\to 0^+}\left(\frac{-3}{8} + \frac{3}{2\sqrt[3]{s^2}}\right) \\ &= -\infty + \infty \end{aligned}$$

L'intégrale est divergente.

e)

$$\begin{aligned} \int_{-1}^1 \frac{1}{\sqrt{1 - x^2}}\, dx &= \int_{-1}^0 \frac{1}{\sqrt{1 - x^2}}\, dx + \int_0^1 \frac{1}{\sqrt{1 - x^2}}\, dx \\ &= \lim_{t\to(-1)^+}\int_t^0 \frac{1}{\sqrt{1 - x^2}}\, dx + \lim_{s\to 1^-}\int_0^s \frac{1}{\sqrt{1 - x^2}}\, dx \\ &= \lim_{t\to(-1)^+}\left(\text{Arc sin } x\,\Big|_t^0\right) + \lim_{s\to 1^-}\left(\text{Arc sin } x\,\Big|_0^s\right) \\ &= \lim_{t\to(-1)^+}(\text{Arc sin } 0 - \text{Arc sin } t) + \lim_{s\to 1^-}(\text{Arc sin } s - \text{Arc sin } 0) \\ &= -\text{Arc sin}(-1) + \text{Arc sin}(1) \\ &= -\left(\frac{-\pi}{2}\right) + \frac{\pi}{2} \\ &= \pi \end{aligned}$$

L'intégrale est convergente.

5

f) $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{e^{x^3}}\,dx = \int_{-\infty}^{0} x^2e^{-x^3}\,dx + \int_{0}^{+\infty} x^2e^{-x^3}\,dx$

$\displaystyle = \lim_{N\to-\infty}\int_{N}^{0} x^2e^{-x^3}\,dx + \lim_{M\to+\infty}\int_{0}^{M} x^2e^{-x^3}\,dx \qquad (u = -x^3)$

$\displaystyle = \lim_{N\to-\infty}\left(\frac{-e^{-x^3}}{3}\Bigg|_{N}^{0}\right) + \lim_{M\to+\infty}\left(\frac{-e^{-x^3}}{3}\Bigg|_{0}^{M}\right)$

$\displaystyle = \lim_{N\to-\infty}\left(\frac{-1}{3} + \frac{1}{3e^{N^3}}\right) + \lim_{M\to+\infty}\left(\frac{-1}{3e^{M^3}} + \frac{1}{3}\right)$

$\displaystyle = \left(\frac{-1}{3} + \infty\right) + \left(0 + \frac{1}{3}\right)$

$= +\infty$

L'intégrale est divergente.

g) $\displaystyle\int_{-1}^{1} \frac{|x|}{x}\,dx = \int_{-1}^{0} \frac{|x|}{x}\,dx + \int_{0}^{1} \frac{|x|}{x}\,dx$

$\displaystyle = \lim_{s\to0^-}\int_{-1}^{s} \frac{(-x)}{x}\,dx + \lim_{t\to0^+}\int_{t}^{1} \frac{(x)}{x}\,dx$

(définition de $|x|$)

$\displaystyle = \lim_{s\to0^-}\int_{-1}^{s} (-1)\,dx + \lim_{t\to0^+}\int_{t}^{1} (1)\,dx$

$\displaystyle = \lim_{s\to0^-}\left(-x\,\Big|_{-1}^{s}\right) + \lim_{t\to0^+}\left(x\,\Big|_{t}^{1}\right)$

$\displaystyle = \lim_{s\to0^-}(-s + 1) + \lim_{t\to0^+}(1 - t)$

$= 2$

L'intégrale est convergente.

h) $\displaystyle\int_{0}^{+\infty} x\sin x\,dx = \lim_{M\to+\infty}\int_{0}^{M} x\sin x\,dx$

$\displaystyle = \lim_{M\to+\infty}(-x\cos x + \sin x)\Big|_{0}^{M}$

$\displaystyle = \lim_{M\to+\infty}(-M\cos M + \sin M)$

Cette limite n'existe pas, d'où l'intégrale est divergente.

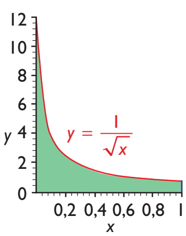
13. a) $\displaystyle A = \int_{0}^{1} \frac{1}{\sqrt{x}}\,dx$

$\displaystyle = \lim_{s\to0^+}\int_{s}^{1} x^{\frac{-1}{2}}\,dx$

$\displaystyle = \lim_{s\to0^+}\left(2\sqrt{x}\,\Big|_{s}^{1}\right)$

$\displaystyle = \lim_{s\to0^+}(2 - 2\sqrt{s})$

$= 2\ \text{u}^2$



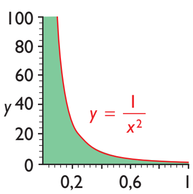
b) $\displaystyle A = \int_{0}^{1} \frac{1}{x^2}\,dx$

$\displaystyle = \lim_{s\to0^+}\int_{s}^{1} \frac{1}{x^2}\,dx$

$\displaystyle = \lim_{s\to0^+}\left(\frac{-1}{x}\Bigg|_{s}^{1}\right)$

$\displaystyle = \lim_{s\to0^+}\left(-1 + \frac{1}{s}\right)$

$= +\infty$



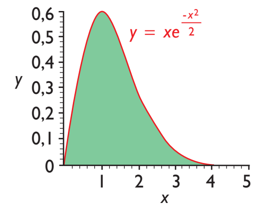
c) $\displaystyle A = \int_{0}^{+\infty} xe^{\frac{-x^2}{2}}\,dx$

$\displaystyle = \lim_{M\to+\infty}\int_{0}^{M} xe^{\frac{-x^2}{2}}\,dx$

$\displaystyle = \lim_{M\to+\infty}\left(-e^{\frac{-x^2}{2}}\,\Big|_{0}^{M}\right)$

$\displaystyle = \lim_{M\to+\infty}\left(-e^{\frac{-M^2}{2}} + 1\right)$

$= 1\ \text{u}^2$



d) Par symétrie :

$\displaystyle A = 2\int_{0}^{2} \frac{x}{\sqrt{4 - x^2}}\,dx$

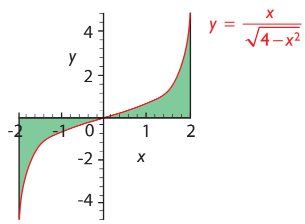
$\displaystyle = 2\lim_{t\to2^-}\int_{0}^{t} \frac{x}{\sqrt{4 - x^2}}\,dx$

$\displaystyle = 2\lim_{t\to2^-}\left(-\sqrt{4 - x^2}\,\Big|_{0}^{t}\right)$

$\displaystyle = 2\lim_{t\to2^-}\left(-\sqrt{4 - t^2} + 2\right)$

$= 2(0 + 2)$

$= 4\ \text{u}^2$



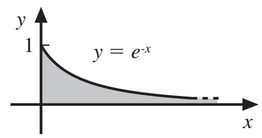
14. a) $\displaystyle A = \int_{0}^{+\infty} e^{-x}\,dx$

$\displaystyle = \lim_{M\to+\infty}\int_{0}^{M} e^{-x}\,dx$

$\displaystyle = \lim_{M\to+\infty}\left(-e^{-x}\,\Big|_{0}^{M}\right)$

$\displaystyle = \lim_{M\to+\infty}(-e^{-M} + e^{0})$

$= 1\ \text{u}^2$



b) i) Autour de l'axe des x.

$\displaystyle V = \int_{0}^{+\infty} \pi(e^{-x})^2\,dx$

$\displaystyle = \lim_{M\to+\infty}\int_{0}^{M} \pi e^{-2x}\,dx$

$\displaystyle = \lim_{M\to+\infty}\left\{\pi\left(\frac{e^{-2x}}{-2}\right)\Bigg|_{0}^{M}\right\}$

$\displaystyle = \lim_{M\to+\infty}\pi\left[\frac{e^{-2M}}{-2} + \frac{1}{2}\right]$

$\displaystyle = \frac{\pi}{2}\ \text{u}^3$

ii) Autour de l'axe des y.

$\displaystyle V = \int_{0}^{+\infty} 2\pi xe^{-x}\,dx$

$\displaystyle = \lim_{M\to+\infty}\int_{0}^{M} 2\pi xe^{-x}\,dx$

Calculons $\int xe^{-x}\,dx = I$.

$u = x$ $du = dx$	$dv = e^{-x}\,dx$ $v = -e^{-x}$

$I = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$

$\displaystyle V = \lim_{M\to+\infty}\left\{2\pi(-xe^{-x} - e^{-x})\,\Big|_{0}^{M}\right\}$

$\displaystyle = \lim_{M\to+\infty} 2\pi\left[\left(\frac{-M}{e^{M}} - \frac{1}{e^{M}}\right) - (0 - e^{0})\right]$

$\displaystyle = 2\pi\ \text{u}^3 \qquad \left(\text{car} \lim_{M\to+\infty}\frac{-M}{e^{M}} \overset{\text{RH}}{=} \lim_{M\to+\infty}\frac{-1}{e^{M}} = 0\right)$

iii) Autour de $y = 1$.

$V = \int_0^1 2\pi x(1 - y)\, dy$

Puisque $y = e^{-x}$, alors $x = -\ln y$.

Ainsi $\lim\limits_{s \to 0^+} \int_s^1 2\pi(-\ln y)(1 - y)\, dy$

Calculons $\int (-\ln y)(1 - y)\, dy = \int (y - 1) \ln y\, dy = I$.

$u = \ln y$	$dv = (y - 1)\, dy$
$du = \dfrac{1}{y}\, dy$	$v = \dfrac{y^2}{2} - y$

$$I = \left(\frac{y^2}{2} - y\right)\ln y - \int \left(\frac{y^2}{2} - y\right)\frac{1}{y}\, dy$$

$$= \left(\frac{y^2}{2} - y\right)\ln y - \frac{y^2}{4} + y + C$$

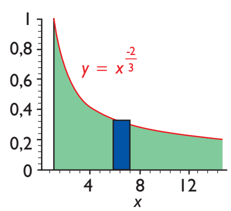
$$V = \lim_{s \to 0^+} \left\{ 2\pi \left[\left(\frac{y^2}{2} - y\right)\ln y - \frac{y^2}{4} + y \right]\Bigg|_s^1 \right\}$$

$$= \lim_{s \to 0^+} 2\pi \left[\frac{3}{4} - \left(\frac{s^2}{2} - s\right)\ln s \right]$$

En appliquant la règle de L'Hospital à $\lim\limits_{s \to 0^+} \dfrac{\ln s}{2s^{-2}}$ et à $\lim\limits_{s \to 0^+} \dfrac{\ln s}{s^{-1}}$, nous trouvons 0,

d'où $V = 2\pi\left(\dfrac{3}{4}\right) = \dfrac{3\pi}{2}$ u³.

15. Base du solide :



a) Volume de la section $= \begin{pmatrix}\text{aire du}\\ \text{carré}\end{pmatrix} \cdot \begin{pmatrix}\text{épaisseur}\\ \text{de la base}\end{pmatrix}$

$$\Delta V = (y^2) \cdot \Delta x$$

$$= \left(\frac{1}{x^{\frac{2}{3}}}\right)^2 \Delta x$$

Ainsi $V = \displaystyle\int_1^{+\infty} \frac{1}{x^{\frac{4}{3}}}\, dx$

$$= \lim_{M \to +\infty} \int_1^M x^{\frac{-4}{3}}\, dx$$

$$= \lim_{M \to +\infty} \left(-3x^{\frac{-1}{3}} \Big|_1^M \right)$$

$$= \lim_{M \to +\infty} \left(\frac{-3}{\sqrt[3]{M}} + 3 \right)$$

$$= 3 \text{ u}^3$$

b) Volume de la section $= \begin{pmatrix}\text{aire du}\\ \text{rectangle}\end{pmatrix} \cdot \begin{pmatrix}\text{épaisseur}\\ \text{de la base}\end{pmatrix}$

$$\Delta V = \left(y \cdot \sqrt{y}\right)\Delta x$$

$$= y^{\frac{3}{2}} \Delta x$$

$$= \left(\frac{1}{x^{\frac{2}{3}}}\right)^{\frac{3}{2}} \Delta x$$

$$= \frac{1}{x}\Delta x$$

Ainsi $V = \displaystyle\int_1^{+\infty} \frac{1}{x}\, dx$

$$= \lim_{M \to +\infty} \int_1^M \frac{1}{x}\, dx$$

$$= \lim_{M \to +\infty} \left(\ln|x| \Big|_1^M \right)$$

$$= \lim_{M \to +\infty} (\ln M)$$

$$= +\infty$$

16. $\dfrac{dQ}{dt} = \dfrac{100t}{(t^2 + 2)^2}$

$$dQ = \frac{100t}{(t^2 + 2)^2}\, dt$$

$$\int dQ = \int \frac{100t}{(t^2 + 2)^2}\, dt$$

Production totale $= 100 \displaystyle\int_0^{+\infty} \frac{t}{(t^2 + 2)^2}\, dt$

$$= 100 \lim_{M \to +\infty} \int_0^M \frac{t}{(t^2 + 2)^2}\, dt$$

$$= 100 \lim_{M \to +\infty} \left(\frac{-1}{2(t^2 + 2)} \Bigg|_0^M \right)$$

$$= 100 \lim_{M \to +\infty} \left(\frac{-1}{2(M^2 + 2)} + \frac{1}{4} \right)$$

$$= 100\left(0 + \frac{1}{4}\right)$$

$$= 25$$

d'où la production totale à partir d'aujourd'hui sera de 25 millions de barils.

Solutionnaire
Problèmes de synthèse

Chapitre 5 (page 303)

1. a) Calculons $I = \int x^2 \ln x \, dx$.

$u = \ln x$, $du = \frac{1}{x}\,dx$ | $dv = x^2\,dx$, $v = \frac{x^3}{3}$

$$I = \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\,dx = \frac{x^3\ln x}{3} - \frac{x^3}{9} + C$$

$$\int_0^1 x^2 \ln x\,dx = \lim_{s\to 0^+}\int_s^1 x^2\ln x\,dx$$
$$= \lim_{s\to 0^+}\left(\frac{x^3\ln x}{3} - \frac{x^3}{9}\right)\Big|_s^1$$
$$= \left(\frac{-1}{9}\right) - \lim_{s\to 0^+}\left(\frac{s^3\ln s}{3} - \frac{s^3}{9}\right) \quad (\text{ind. } 0\cdot(-\infty))$$
$$= \frac{-1}{9} - \frac{1}{3}\lim_{s\to 0^+}\frac{\ln s}{s^{-3}} \quad \left(\text{ind. } \frac{-\infty}{+\infty}\right)$$
$$\overset{\text{RH}}{=} \frac{-1}{9} - \frac{1}{3}\lim_{s\to 0^+}\frac{\frac{1}{s}}{-3s^{-4}}$$
$$= \frac{-1}{9} + \frac{1}{9}\lim_{s\to 0^+} s^3$$
$$= \frac{-1}{9}, \text{ donc l'intégrale est convergente.}$$

b) Calculons $I = \int e^x \sin x\,dx$.

$u = e^x$, $du = e^x\,dx$ | $dv = \sin x\,dx$, $v = -\cos x$

$$I = -e^x\cos x + \int e^x \cos x\,dx$$

$u = e^x$, $du = e^x\,dx$ | $dv = \cos x\,dx$, $v = \sin x$

$$I = -e^x\cos x + e^x \sin x - I$$
$$2I = -e^x\cos x + e^x\sin x + C_1$$
$$I = \frac{e^x(\sin x - \cos x)}{2} + C$$
$$\int_0^{+\infty} e^x\sin x\,dx = \lim_{M\to+\infty}\int_0^M e^x \sin x\,dx$$
$$= \lim_{M\to+\infty}\left[\frac{e^x(\sin x - \cos x)}{2}\right]\Big|_0^M$$
$$= \lim_{M\to+\infty}\left[\frac{e^M(\sin M - \cos M)}{2}\right] + \frac{1}{2}$$

Puisque cette limite n'existe pas, l'intégrale est divergente.

c) Calculons $I = \int e^x \cos x\,dx$.

$u = e^x$, $du = e^x\,dx$ | $dv = \cos x\,dx$, $v = \sin x$

$$\int e^x\cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

$u = e^x$, $du = e^x\,dx$ | $dv = \sin x\,dx$, $v = -\cos x$

$$I = e^x\sin x + e^x\cos x - I$$
$$2I = e^x\sin x + e^x \cos x + C_1$$
$$I = \frac{e^x(\sin x + \cos x)}{2} + C$$
$$\int_{-\infty}^0 e^x\cos x\,dx = \lim_{N\to-\infty}\int_N^0 e^x\cos x\,dx$$
$$= \lim_{N\to-\infty}\left[\frac{e^x(\sin x + \cos x)}{2}\right]\Big|_N^0$$
$$= \frac{1}{2} - \lim_{N\to-\infty}\frac{e^N(\sin N + \cos N)}{2}$$
$$= \frac{1}{2} \quad \left(\text{car } |\sin N + \cos N| < 2 \text{ et } \lim_{N\to-\infty} e^N = 0\right)$$

donc l'intégrale est convergente.

d)
$$\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx$$
$$= \int \frac{-1}{u}\,du \qquad (u = e^{-x}+1)$$
$$= -\ln|u| + C$$
$$= -\ln(e^{-x}+1) + C$$
$$\int_0^{+\infty}\frac{1}{1+e^x}\,dx = \lim_{M\to+\infty}\int_0^M \frac{1}{1+e^x}\,dx$$
$$= \lim_{M\to+\infty}\left(-\ln(e^{-x}+1)\right)\Big|_0^M$$
$$= \lim_{M\to+\infty}\left(-\ln(e^{-M}+1) + \ln 2\right)$$
$$= \ln 2$$

donc l'intégrale est convergente.

e)
$$\int \frac{1}{1-\sin x}\,dx = \int \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx$$
$$= \int \frac{1+\sin x}{\cos^2 x}\,dx$$
$$= \int (\sec^2 x + \sec x \tan x)\,dx$$
$$= \tan x + \sec x + C$$

5

$$\int_0^{\frac{\pi}{2}} \frac{1}{1-\sin x}\,dx = \lim_{t\to\left(\frac{\pi}{2}\right)^-} \int_0^t \frac{1}{1-\sin x}\,dx$$
$$= \lim_{t\to\left(\frac{\pi}{2}\right)^-} (\tan x + \sec x)\Big|_0^t$$
$$= \lim_{t\to\left(\frac{\pi}{2}\right)^-} \left[(\tan t + \sec t) - 1\right]$$
$$= +\infty$$

donc l'intégrale est divergente.

f) $$I = \int \frac{x^{\frac{2}{5}}+1}{x^{\frac{2}{5}}-4}\,dx = 5\int \frac{u^4(u^2+1)}{u^2-4}\,du \qquad \left(u = x^{\frac{1}{5}}\right)$$

Or, $$\frac{u^4(u^2+1)}{u^2-4} = u^4 + 5u^2 + 20 + \frac{80}{u^2-4}$$
$$= u^4 + 5u^2 + 20 + \frac{20}{u-2} - \frac{20}{u+2}$$

$$I = 5\int\left(u^4 + 5u^2 + 20 + \frac{20}{u-2} - \frac{20}{u+2}\right)du$$
$$= 5\left[\frac{u^5}{5} + \frac{5u^3}{3} + 20u + 20\ln|u-2| - 20\ln|u+2|\right] + C$$
$$= 5\left[\frac{x}{5} + \frac{5x^{\frac{3}{5}}}{3} + 20x^{\frac{1}{3}} + 20\ln\left|\frac{x^{\frac{1}{3}}-2}{x^{\frac{1}{3}}+2}\right|\right] + C$$

$$\int_0^1 \frac{x^{\frac{2}{5}}+1}{x^{\frac{2}{5}}-4}\,dx = 5\left[\frac{x}{5} + \frac{5x^{\frac{3}{5}}}{3} + 20x^{\frac{1}{5}} + 20\ln\left|\frac{x^{\frac{1}{5}}-2}{x^{\frac{1}{5}}+2}\right|\right]\Bigg|_0^1$$
$$= \frac{328}{3} - 100\ln 3$$

donc l'intégrale est convergente.

g) $$\int \frac{x-2}{\sqrt{x-1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du \qquad (u = x-1)$$
$$= \int\left(\sqrt{u} - \frac{1}{\sqrt{u}}\right)du$$
$$= \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + C$$
$$= \frac{2}{3}(x-1)^{\frac{3}{2}} - 2\sqrt{x-1} + C$$

$$\int_1^2 \frac{x-2}{\sqrt{x-1}}\,dx = \lim_{s\to 1^+}\int_s^2 \frac{x-2}{\sqrt{x-1}}\,dx$$
$$= \lim_{s\to 1^+}\left[\frac{2}{3}(x-1)^{\frac{3}{2}} - 2\sqrt{x-1}\right]\Bigg|_s^2$$
$$= \lim_{s\to 1^+}\left[\left(\frac{2}{3}-2\right) - \left(\frac{2}{3}(s-1)^{\frac{3}{2}} - 2\sqrt{s-1}\right)\right]$$
$$= \frac{-4}{3}$$

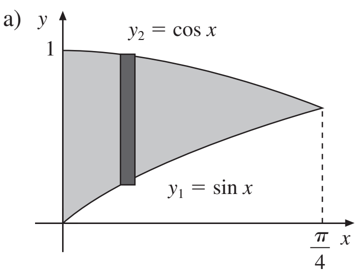
donc l'intégrale est convergente.

h) $$\int \frac{1}{\sin\theta\cos\theta\sqrt{\tan^2\theta-1}}\,d\theta = \int \frac{\sec^2\theta}{\tan\theta\sqrt{\tan^2\theta-1}}\,d\theta$$
$$= \int \frac{1}{u\sqrt{u^2-1}}\,du \quad (u = \tan\theta)$$
$$= \text{Arc sec}\,u + C$$
$$= \text{Arc sec}\,(\tan\theta) + C$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin\theta\cos\theta\sqrt{\tan^2\theta-1}}\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{8}} \frac{1}{\sin\theta\cos\theta\sqrt{\tan^2\theta-1}}\,d\theta + \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{1}{\sin\theta\cos\theta\sqrt{\tan^2\theta-1}}\,d\theta$$
$$= \lim_{s\to\left(\frac{\pi}{4}\right)^+}\int_s^{\frac{3\pi}{8}} \ldots\,d\theta + \lim_{t\to\left(\frac{\pi}{2}\right)^-}\int_{\frac{3\pi}{8}}^t \ldots\,d\theta$$
$$= \lim_{s\to\left(\frac{\pi}{4}\right)^+} \text{Arc sec}\,(\tan\theta)\Big|_s^{\frac{3\pi}{8}} + \lim_{t\to\left(\frac{\pi}{2}\right)^-} \text{Arc sec}\,(\tan\theta)\Big|_{\frac{3\pi}{8}}^t$$
$$= \lim_{s\to\left(\frac{\pi}{4}\right)^+}\left[\text{Arc sec}\left(\tan\frac{3\pi}{8}\right) - \text{Arc sec}\,(\tan s)\right] + \lim_{t\to\left(\frac{\pi}{2}\right)^-}\left[\text{Arc sec}\,(\tan t) - \text{Arc sec}\left(\tan\frac{3\pi}{8}\right)\right]$$
$$= -\text{Arc sec}\left(\tan\frac{\pi}{4}\right) + \frac{\pi}{2}$$
$$= -\text{Arc sec}\,(1) + \frac{\pi}{2} = \frac{\pi}{2}$$

donc l'intégrale est convergente.

2. a)



$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx$$
$$= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}}$$
$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) - (\sin 0 + \cos 0)$$
$$= \left(\sqrt{2} - 1\right)\text{u}^2$$

b) i) Le volume de révolution autour de l'axe des x est donné par

$$V_{0X} = V_1 - V_2$$
$$= \pi\int_0^{\frac{\pi}{4}} (\cos x)^2\,dx - \pi\int_0^{\frac{\pi}{4}} (\sin x)^2\,dx$$
$$= \pi\int_0^{\frac{\pi}{4}} (\cos^2 x - \sin^2 x)\,dx$$
$$= \pi\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$
$$= \pi\left(\frac{\sin 2x}{2}\right)\Bigg|_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{2}\,\text{u}^3$$

5

ii) Le volume de révolution autour de l'axe des y est donné par

$$V_{0Y} = 2\pi\int_0^{\frac{\pi}{4}} x(\cos x - \sin x)\,dx$$

Calculons $I = \int x(\cos x - \sin x)\,dx$.

$u = x$ $du = dx$	$dv = (\cos x - \sin x)\,dx$ $v = (\sin x + \cos x)$

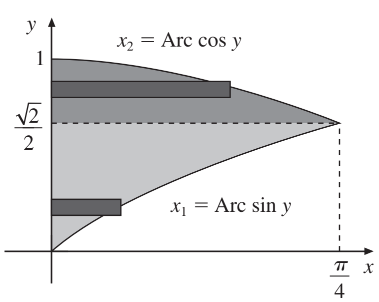
$$\begin{aligned} I &= x(\sin x + \cos x) - \int(\sin x + \cos x)\,dx \\ &= x(\sin x + \cos x) + (\cos x - \sin x) + C \end{aligned}$$

$$\begin{aligned} V_{0Y} &= 2\pi\left[x(\sin x + \cos x) + (\cos x - \sin x)\right]\Big|_0^{\frac{\pi}{4}} \\ &= \left(\frac{\sqrt{2}\pi^2}{2} - 2\pi\right)\text{u}^3 \end{aligned}$$

c) i) Lorsque chaque section est perpendiculaire à l'axe des x,

$$\begin{aligned} V &= \int_0^{\frac{\pi}{4}} (\cos x - \sin x)^2\,dx \\ &= \int_0^{\frac{\pi}{4}} (\cos^2 x - 2\sin x\cos x + \sin^2 x)\,dx \\ &= \int_0^{\frac{\pi}{4}} (1 - 2\sin x\cos x)\,dx \\ &= (x - \sin^2 x)\Big|_0^{\frac{\pi}{4}} \\ &= \left(\frac{\pi}{4} - \frac{1}{2}\right)\text{u}^3 \end{aligned}$$

ii)



Lorsque chaque section est perpendiculaire à l'axe des y, le volume est donné par

$$V = V_1 + V_2$$

où $V_1 = \int_0^{\frac{\sqrt{2}}{2}} (\text{Arc}\sin y)^2\,dy$ et

$V_2 = \int_{\frac{\sqrt{2}}{2}}^1 (\text{Arc}\cos y)^2\,dy$

Déterminons d'abord $\int (\text{Arc}\sin y)^2\,dy$, que nous notons I.

$u = (\text{Arc}\sin y)^2$ $du = \dfrac{2(\text{Arc}\sin y)}{\sqrt{1-y^2}}\,dy$	$dv = dy$ $v = y$

$$I = y(\text{Arc}\sin y)^2 - 2\int \frac{(\text{Arc}\sin y)y}{\sqrt{1-y^2}}\,dy$$

$u = \text{Arc}\sin y$ $du = \dfrac{1}{\sqrt{1-y^2}}\,dy$	$dv = \dfrac{y}{\sqrt{1-y^2}}\,dy$ $v = -\sqrt{1-y^2}$

$$\begin{aligned} I &= (\text{Arc}\sin y)^2 - 2\left[-\sqrt{1-y^2}\,\text{Arc}\sin y + \int 1\,dy\right] \\ &= (\text{Arc}\sin y)^2 + 2\sqrt{1-y^2}\,\text{Arc}\sin y - 2y + C \end{aligned}$$

$$\begin{aligned} V_1 &= \int_0^{\frac{\sqrt{2}}{2}} (\text{Arc}\sin y)^2\,dy \\ &= \left[y(\text{Arc}\sin y)^2 + 2\sqrt{1-y^2}\,\text{Arc}\sin y - 2y\right]\Big|_0^{\frac{\sqrt{2}}{2}} \\ &= \left[\frac{\sqrt{2}}{2}\left(\frac{\pi}{4}\right)^2 + \sqrt{2}\left(\frac{\pi}{4}\right) - \sqrt{2}\right] \end{aligned}$$

De façon analogue,

$$\begin{aligned} V_2 &= \int_{\frac{\sqrt{2}}{2}}^1 (\text{Arc}\cos y)^2\,dy \\ &= \left[y(\text{Arc}\cos y)^2 - 2\sqrt{1-y^2}\,\text{Arc}\cos y - 2y\right]\Big|_{\frac{\sqrt{2}}{2}}^1 \\ &= \left[-2 - \frac{\sqrt{2}}{2}\left(\frac{\pi}{4}\right)^2 + \sqrt{2}\left(\frac{\pi}{4}\right) + \sqrt{2}\right] \end{aligned}$$

d'où $V = V_1 + V_2$

$$= \left(\frac{\sqrt{2}\pi}{2} - 2\right)\text{u}^3$$

3. a) $f(x) = a - \dfrac{6x}{x^2 - b^2}$, ainsi

$2 = a - \dfrac{6(0)}{0 - b^2}$ (car $f(0) = 2$)

donc $a = 2$;
f n'est pas définie en $x = -2$ et en $x = 2$
donc $b = -2$ ou $b = 2$
d'où $a = 2$, $b = -2$ ou $b = 2$

b)

$$\begin{aligned} f(x) &= 0 \\ 2 - \frac{6x}{x^2 - 4} &= 0 \\ \frac{-6x}{x^2 - 4} &= -2 \\ -6x &= -2x^2 + 8 \\ 2x^2 - 6x - 8 &= 0 \\ 2(x^2 - 3x - 4) &= 0 \\ 2(x + 1)(x - 4) &= 0 \end{aligned}$$

d'où $c = -1$ et $d = 4$

c) i) $\pi\displaystyle\int_{-1}^0 \left(2 - \frac{6x}{x^2 - 4}\right)^2 dx = \pi\int_{-1}^0 \left(4 - \frac{24x}{x^2 - 4} + \frac{36x^2}{(x^2 - 4)^2}\right)dx$

où

$$\begin{aligned} \frac{x^2}{(x^2 - 4)^2} &= \frac{A}{x - 2} + \frac{B}{(x - 2)^2} + \frac{C}{x + 2} + \frac{D}{(x + 2)^2} \\ &= \frac{1}{8(x - 2)} + \frac{1}{4(x - 2)^2} - \frac{1}{8(x + 2)} + \frac{1}{4(x + 2)^2} \end{aligned}$$

5

$$\pi\int_{-1}^{0}\left(2-\frac{6x}{x^2-4}\right)^2 dx = \pi\left(\frac{15}{2}\ln(3)+10-24\ln(2)\right)$$
$$\approx 5{,}039\ \text{u}^3$$

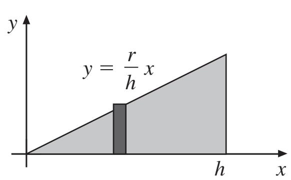
ii) $$2\pi\int_{-1}^{0}\left(-x\left(2-\frac{6x}{x^2-4}\right)\right)dx$$
$$= 2\pi\int_{-1}^{0}\left(-2x+\frac{6x^2}{x^2-4}\right)dx$$
$$= 2\pi\int_{-1}^{0}\left(-2x+6\left(1+\frac{4}{x^2-4}\right)\right)dx$$
$$= 2\pi\int_{-1}^{0}\left(-2x+6\left(1+\frac{1}{x-2}-\frac{1}{x+2}\right)\right)dx$$
$$= 2\pi(7-6\ln 3)$$
$$\approx 2{,}565\ \text{u}^3$$

d) i) $$\int_{4}^{16}\left(2-\frac{6x}{x^2-4}\right)dx = \left(2x-3\ln\left|x^2-4\right|\right)\Big|_4^{16}$$
$$= (32-3\ln 252)-(8-3\ln 12)$$
$$= 24+3\ln\left(\frac{12}{252}\right)$$
$$\approx 14{,}866\ \text{u}^2$$

ii) $$\int_{4}^{+\infty}\left(2-\left(2-\frac{6x}{x^2-4}\right)\right)dx = \int_{4}^{+\infty}\frac{6x}{x^2-4}dx$$
$$= \lim_{M\to+\infty}\int_{4}^{M}\frac{6x}{x^2-4}dx$$
$$= \lim_{M\to+\infty}3\ln\left|x^2-4\right|\Big|_4^{M}$$
$$= \lim_{M\to+\infty}(3\ln M-3\ln 12)$$
$$= +\infty$$

d'où l'aire est infinie.

4. a) La région délimitée par $y = \frac{r}{h}x$, $y = 0$ et $x = h$.



b) $$V = \int_0^h \pi y^2\,dx$$
$$= \pi\int_0^h \frac{r^2x^2}{h^2}dx$$
$$= \frac{\pi r^2}{h^2}\frac{x^3}{3}\Big|_0^h$$
$$= \frac{\pi r^2 h}{3}\ \text{u}^3$$

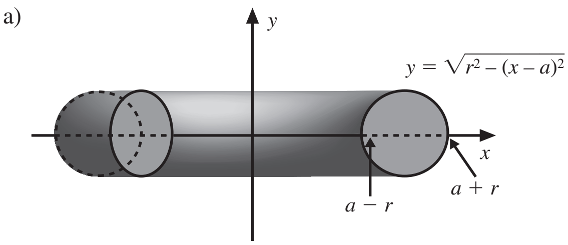
c) Déterminons l'équation de la droite passant par $(0, r)$ et (h, R).

Puisque $m = \frac{R-r}{h}$, et que la droite passe par $(0, r)$,

nous avons $y = \frac{R-r}{h}x + r$

$$V = \int_0^h \pi y^2\,dx$$
$$= \pi\int_0^h\left(\frac{R-r}{h}x+r\right)^2 dx$$
$$= \frac{\pi h}{R-r}\int_r^R u^2\,du \qquad \left(u = \frac{R-r}{h}x+r\right)$$
$$= \frac{\pi h}{(R-r)}\left(\frac{u^3}{3}\right)\Big|_r^R$$
$$= \frac{\pi h}{3(R-r)}(R^3-r^3)$$
$$= \frac{\pi h}{3}(R^2+Rr+r^2)\ \text{u}^3$$

5. a)



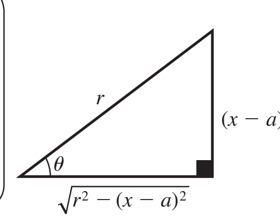
Le volume $V_{\frac{1}{2}}$ de la partie située au-dessus de l'axe des x est donné par

$$V_{\frac{1}{2}} = 2\pi\int_{a-r}^{a+r} xy\,dx$$

et correspond à la moitié du volume V cherché.

$$V = 2\left[2\pi\int_{a-r}^{a+r} x\sqrt{r^2-(x-a)^2}\,dx\right]$$
$$\left(\text{car } y = \sqrt{r^2-(x-a)^2}\right)$$

$$(x-a)^2 = r^2\sin\theta$$
$$x-a = r\sin\theta$$
$$x = a+r\sin\theta$$
$$dx = r\cos\theta\,d\theta$$
$$\theta = \text{Arc}\sin\left(\frac{x-a}{r}\right)$$



$$I = \int x\sqrt{r^2(x-a)^2}\,dx$$
$$= \int(a+r\sin\theta)\sqrt{r^2-r^2\sin^2\theta}\,r\cos\theta\,d\theta$$
$$= \int(a+r\sin\theta)r^2\cos^2\theta\,d\theta$$
$$= ar^2\int\cos^2\theta\,d\theta + r^3\int\sin\theta\cos^2\theta\,d\theta$$
$$= ar^2\left(\frac{\sin\theta\cos\theta+\theta}{2}\right)-\frac{r^3\cos^3\theta}{3}+C$$

$$I = \frac{ar^2}{2}\left[\frac{(x-a)\sqrt{r^2-(x-a)^2}}{r^2}+\text{Arc}\sin\left(\frac{x-a}{4}\right)\right]-\frac{r^3}{3}\left(\frac{\sqrt{r^2-(x-a)^2}}{r}\right)^3+C$$

$$V = 4\pi\left[\frac{a(x-a)\sqrt{r^2-(x-a)^2}}{2} + \frac{ar^2}{2}\,\text{Arc sin}\left(\frac{x-a}{r}\right) - \frac{\left(\sqrt{r^2-(x-a)^2}\right)^3}{3}\right]\Bigg|_{a-r}^{a+r}$$

$$= 2\pi^2 ar^2 \text{ u}^3$$

L'aire $A_{\frac{1}{2}}$ de la partie située au-dessus de l'axe des x est donnée par $A_{\frac{1}{2}} = 2\pi\int_0^{\pi} x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

et correspond à la moitié de l'aire A cherchée.

$x = a + r\cos t$, ainsi $\frac{dx}{dt} = \text{-}r\sin t$

$y = r\sin t$, ainsi $\frac{dy}{dt} = r\cos t$

$$\text{Ainsi } A = 2\left[2\pi\int_0^{\pi}(a + r\cos t)\sqrt{(\text{-}r\sin t)^2 + (r\cos t)^2}\,dt\right]$$

$$= 4\pi r\int_0^{\pi}(a + r\cos t)\,dt$$

$$= 4\pi r\,[at + r\sin t]\Big|_0^{\pi}$$

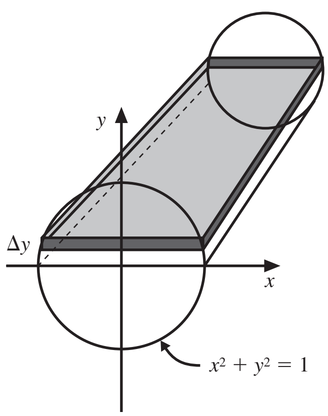
$$= 4\pi r\,[a\pi]$$

$$= 4\pi^2 ar \text{ u}^2$$

b) Sachant que $V = 2\pi^2 ar^2$,
en posant $r = 2$ et $a = 3$, nous obtenons $V_1 = 24\pi^2$ u³,
et en posant $r = 1$ et $a = 10$, nous obtenons
$V_2 = 20\pi^2$ u³,
d'où $V_1 > V_2$.

c) Nous cherchons a, tel que $2\pi^2\,a(1)^2 = 24\pi^2$,
d'où $a = 12$.

6.



$$V = 12\int_{-1}^{\frac{1}{2}} 2x\,dy$$

$$= 24\int_{-1}^{\frac{1}{2}}\sqrt{1-y^2}\,dy$$

$$= 24\left[\frac{y\sqrt{1-y^2}}{2} + \frac{1}{2}\text{Arc sin } y\right]\Bigg|_{-1}^{\frac{1}{2}} \quad \text{(formule 3, page 472)}$$

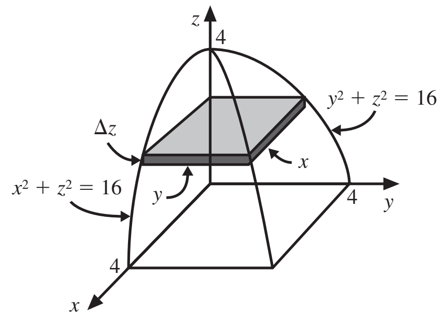
$$= 12\left[\left(\frac{1}{2}\sqrt{\frac{3}{4}} + \text{Arc sin}\left(\frac{1}{2}\right)\right) - (0 + \text{Arc sin}\,(\text{-}1))\right]$$

$$= 12\left[\frac{\sqrt{3}}{4} + \frac{\pi}{6} + \frac{\pi}{2}\right]$$

$$\approx 30{,}329$$

d'où environ 30 329 litres.

7.



Calculons le volume dans le premier octant et multiplions ce volume par 8 pour obtenir le volume cherché.

$$\Delta V \approx (\text{aire d'une section})\cdot(\text{épaisseur de la section})$$

$$\approx xy\,\Delta z$$

$$\approx x^2\,\Delta z \qquad (\text{car } x = y)$$

$$\approx (16 - z^2)\,dz \qquad (\text{car } x^2 = 16 - z^2)$$

$$\text{Ainsi } V = 8\int_0^4 (16 - z^2)\,dz$$

$$= 8\left(16z - \frac{z^3}{3}\right)\Bigg|_0^4$$

$$= \frac{1024}{3} \text{ u}^3$$

8. En posant $x = 0$ et $y = 4$, nous obtenons

$4 = \frac{a(e^0 + e^0)}{2}$, d'où $a = 4$.

Ainsi $y = 2\left(e^{\frac{x}{4}} + e^{\frac{-x}{4}}\right)$ et $\frac{dy}{dx} = \frac{e^{\frac{x}{4}} - e^{\frac{-x}{4}}}{2}$

$$L = \int_{-3,9}^{6,3}\sqrt{1 + \left(\frac{e^{\frac{x}{4}} - e^{\frac{-x}{4}}}{2}\right)^2}\,dx$$

$$= \int_{-3,9}^{6,3}\sqrt{1 + \frac{e^{\frac{x}{2}} - 2 + e^{\frac{-x}{2}}}{4}}\,dx$$

$$= \int_{-3,9}^{6,3}\sqrt{\frac{e^{\frac{x}{2}} + 2 + e^{\frac{-x}{2}}}{4}}\,dx$$

$$= \int_{-3,9}^{6,3}\frac{e^{\frac{x}{4}} + e^{\frac{-x}{4}}}{2}\,dx$$

$$= 2\left(e^{\frac{x}{4}} - e^{\frac{-x}{4}}\right)\Big|_{-3,9}^{6,3}$$

$$= 2\left(e^{\frac{6,3}{4}} - e^{\frac{-6,3}{4}}\right) - 2\left(e^{\frac{-3,9}{4}} - e^{\frac{3,9}{4}}\right)$$

$$\approx 13{,}8 \text{ m}$$

9. a) $L_1 = \int_0^1\sqrt{1 + (3x^2)^2}\,dx$

$$\approx \frac{1-0}{3(4)}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + f(1)\right]$$

$$\approx \frac{1}{12}\left[1 + 4\sqrt{\frac{265}{256}} + 2\sqrt{\frac{25}{16}} + 4\sqrt{\frac{985}{256}} + \sqrt{10}\right]$$

$$\approx 1{,}548 \text{ km}$$

$$L \approx 2L_1 + 1$$

$$\approx 4{,}096 \text{ km}$$

b) Soit le point milieu du segment BC le point d'origine d'un système d'axes. Ainsi, les coordonnées des points A et D sont $A\left(\frac{-3}{2}, -1\right)$ et $D\left(\frac{3}{2}, 1\right)$.

D'où la distance L joignant A à D est donnée par

$$L = \sqrt{(1-(-1))^2 + \left(\frac{3}{2} - \left(\frac{-3}{2}\right)\right)^2} = \sqrt{13} \approx 3{,}606 \text{ km.}$$

10. Calculons le quart de la longueur totale parcourue par le petit cercle.

$$L_{\frac{1}{4}} = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{\left[\frac{d}{d\theta}(5\cos\theta - \cos 5\theta)\right]^2 + \left[\frac{d}{d\theta}(5\sin\theta - \sin 5\theta)\right]^2}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{(-5\sin\theta + 5\sin 5\theta)^2 + (5\cos\theta - 5\cos 5\theta)^2}\, d\theta$$

$$= \sqrt{50}\int_0^{\frac{\pi}{2}} \sqrt{1 - \sin\theta\sin 5\theta - \cos\theta\cos 5\theta}\, d\theta$$

$$= 5\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{1}{2}(\cos 4\theta - \cos 6\theta) - \frac{1}{2}(\cos 4\theta + \cos 6\theta)}\, d\theta$$

$$= 5\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1 - \cos 4\theta}\, d\theta$$

$$= 5\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{2\sin^2(2\theta)}\, d\theta$$

$$= 10\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta$$

$$= 10\left(\frac{-\cos(2\theta)}{2}\Bigg|_0^{\frac{\pi}{2}}\right)$$

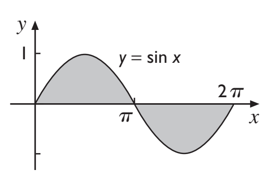
$$= 10\left(\frac{-\cos\pi + \cos 0}{2}\right) = 10$$

Ainsi $L_{\frac{1}{4}} = 10$ u

D'où la longueur totale L est donnée par $L = 40$ u.

11. a) $A = 2\int_0^{\pi} \sin x\, dx$

$$= -2\cos x\,\Big|_0^{\pi}$$
$$= -2\cos\pi + 2\cos 0$$
$$= 4 \text{ u}^2$$



b) $V = 4\int_0^{\frac{\pi}{2}} \pi\sin^2 x\, dx$

$$= 4\pi\left\{\frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]\right\}\Bigg|_0^{\frac{\pi}{2}} \quad \text{(formule 43, page 474)}$$

$$= 2\pi\left[\frac{\pi}{2} - 0\right]$$

$$= \pi^2 \text{ u}^3$$

c) $S = 2\int_0^{\pi} 2\pi\sin x\sqrt{1+\cos^2 x}\, dx$

Calculons $\int \sin x\sqrt{1+\cos^2 x}\, dx = I$.

$$I = -\int \sqrt{1+u^2}\, du \qquad (u = \cos x)$$

$$= -\left[\frac{u}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left|u + \sqrt{1+u^2}\right|\right] + C$$

(formule 13, page 473)

$$= -\left[\frac{\cos x}{2}\sqrt{1+\cos^2 x} + \frac{1}{2}\ln\left|\cos x + \sqrt{1+\cos^2 x}\right|\right] + C$$

Ainsi

$$S = -4\pi\left[\frac{\cos x}{2}\sqrt{1+\cos^2 x} + \frac{1}{2}\ln\left|\cos x + \sqrt{1+\cos^2 x}\right|\right]\Bigg|_0^{\pi}$$

$$= 2\pi\left[2\sqrt{2} + \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)\right]$$

$$\approx 28{,}85 \text{ u}^2$$

d) $L = 2\int_0^{\pi} \sqrt{1+\cos^2 x}\, dx$

$$\approx 2\,\frac{(\pi - 0)}{2(4)}\left[f(0) + 2f\left(\frac{\pi}{4}\right) + 2f\left(\frac{\pi}{2}\right) + 2f\left(\frac{3\pi}{4}\right) + f(\pi)\right]$$

$$\approx \frac{\pi}{4}\left[\sqrt{2} + 2\sqrt{\frac{3}{2}} + 2\sqrt{1} + 2\sqrt{\frac{3}{2}} + \sqrt{2}\right]$$

$$\approx 7{,}64 \text{ u}$$

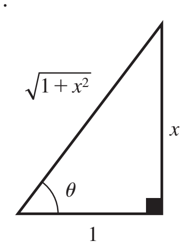
12. a) Autour de l'axe des x.

$$V = \int_0^{+\infty} \pi\left(\frac{1}{x^2+1}\right)^2 dx$$

$$= \lim_{M\to+\infty}\int_0^{M} \pi\left(\frac{1}{x^2+1}\right)^2 dx$$

Calculons $\int\left(\frac{1}{x^2+1}\right)^2 dx = I$.

$$x^2 = \tan^2\theta$$
$$x = \tan\theta$$
$$dx = \sec^2\theta\, d\theta$$
$$\theta = \text{Arc}\tan x$$



$$I = \int \frac{1}{(\tan^2\theta + 1)^2}\sec^2\theta\, d\theta$$

$$= \int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + C \quad \text{(formule 44, page 474)}$$

$$= \frac{1}{2}[\theta + \sin\theta\cos\theta] + C$$

$$= \frac{1}{2}\left(\text{Arc}\tan x + \frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}\right) + C$$

5

$$\text{Donc } V = \lim_{M\to+\infty} \pi\left\{\left[\frac{1}{2}\left(\text{Arc}\tan x + \frac{x}{1+x^2}\right)\right]\Bigg|_0^M\right\}$$

$$= \lim_{M\to+\infty} \pi\left[\frac{1}{2}\left(\text{Arc}\tan M + \frac{M}{1+M^2}\right) - \frac{1}{2}(0)\right]$$

$$= \frac{\pi}{2}\lim_{M\to+\infty} \text{Arc}\tan M + \frac{\pi}{2}\lim_{M\to+\infty}\frac{M}{1+M^2}$$

$$= \frac{\pi}{2}\cdot\frac{\pi}{2} + 0$$

$$\text{d'où } V = \frac{\pi^2}{4}\text{ u}^3$$

b) Autour de l'axe des y.

$$V = \int_0^{+\infty} 2\pi x\left(\frac{1}{x^2+1}\right)dx$$

$$= \lim_{M\to+\infty}\int_0^M 2\pi x\left(\frac{1}{x^2+1}\right)dx$$

$$= \lim_{M\to+\infty}\left\{2\pi\left[\frac{1}{2}\ln(x^2+1)\right]\Bigg|_0^M\right\}$$

$$= \lim_{M\to+\infty} 2\pi\left[\left(\frac{1}{2}\ln(M^2+1)\right) - \frac{1}{2}\ln 1\right]$$

$$= +\infty$$

d'où le volume est infini.

13. a) $$V_{0X} = \pi\int_1^{+\infty}(f(x))^2\,dx$$

$$= \pi\int_1^{+\infty}\left(\frac{1}{x^p}\right)^2 dx$$

$$= \pi\lim_{M\to+\infty}\int_1^M \frac{1}{x^{2p}}\,dx$$

Si $2p \leq 1$, c'est-à-dire $p \leq \dfrac{1}{2}$,

alors $x^{2p} \leq x$

$$\frac{1}{x^{2p}} \geq \frac{1}{x} \geq 0$$

$$\text{Or } \int_1^{+\infty}\frac{1}{x}\,dx = \lim_{M\to+\infty}\int_1^M \frac{1}{x}\,dx$$

$$= \lim_{M\to+\infty}\left(\ln|x|\,\Big|_1^M\right)$$

$$= \lim_{M\to+\infty}(\ln M - \ln 1)$$

$$= +\infty$$

Donc $\displaystyle\int_1^{+\infty}\frac{1}{x^{2p}}\,dx$ est divergente pour $2p \leq 1$.

(théorème 5.4, page 297)

$$\text{Si } 2p > 1,\ \lim_{M\to+\infty}\int_1^M \frac{1}{x^{2p}}\,dx = \lim_{M\to+\infty}\int_1^M x^{-2p}\,dx$$

$$= \lim_{M\to+\infty}\left(\frac{x^{-2p+1}}{-2p+1}\Bigg|_1^M\right)$$

$$= \lim_{M\to+\infty}\left(\frac{1}{(-2p+1)M^{2p-1}} - \frac{1}{-2p+1}\right)$$

$$= \left(0 - \frac{1}{-2p+1}\right) \quad (\text{car } (2p-1) > 0)$$

$$= \frac{1}{2p-1}$$

$$\text{donc, si } p > \frac{1}{2},\ V_{0X} = \pi\lim_{M\to+\infty}\int_1^{+\infty}\frac{1}{x^{2p}}\,dx$$

$$\text{d'où } V_{0X} = \frac{\pi}{2p-1}\text{ u}^3$$

b) $$V_{0Y} = 2\pi\int_1^{+\infty} x\,f(x)\,dx$$

$$= 2\pi\int_1^{+\infty} x\left(\frac{1}{x^p}\right)dx$$

$$= 2\pi\lim_{M\to+\infty}\int_1^M \frac{1}{x^{p-1}}\,dx$$

Si $(p-1) \leq 1$, c'est-à-dire $p \leq 2$,

alors $x^{p-1} \leq x$

$$\frac{1}{x^{p-1}} \geq \frac{1}{x} \geq 0$$

Donc $\displaystyle\int_1^{+\infty}\frac{1}{x^{p-1}}\,dx$ est divergente.

(théorème 5.4, page 297)

$$\text{Si } (p-1) > 1,\ \lim_{M\to+\infty}\int_1^M \frac{1}{x^{p-1}}\,dx$$

$$= \lim_{M\to+\infty}\int_1^M x^{1-p}\,dx$$

$$= \lim_{M\to+\infty}\left(\frac{x^{2-p}}{2-p}\Bigg|_1^M\right)$$

$$= \lim_{M\to+\infty}\left(\frac{1}{(2-M)x^{p-2}} - \frac{1}{2-p}\right)$$

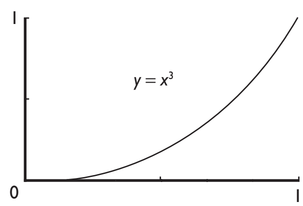
$$= \left(0 - \frac{1}{2-p}\right) \quad (\text{car } (p-2) > 0)$$

$$= \frac{1}{p-2}$$

$$\text{donc, si } p > 2,\ V_{0Y} = 2\pi\lim_{M\to+\infty}\int_1^M \frac{1}{x^{p-1}}\,dx$$

$$\text{d'où } V_{0Y} = \frac{2\pi}{p-2}\text{ u}^3$$

14. a)

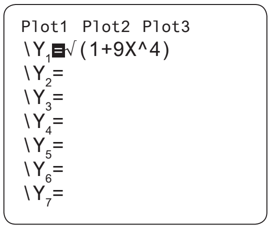


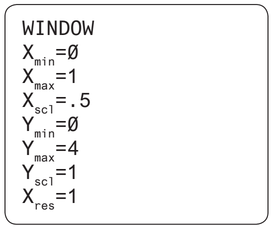
$$\frac{dy}{dx} = 3x^2$$

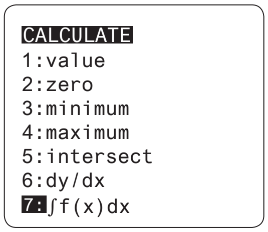
$$L = \int_0^1 \sqrt{1+(3x^2)^2}\,dx$$

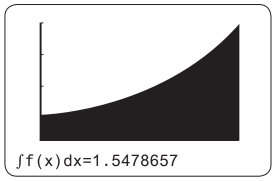
$$= \int_0^1 \sqrt{1+9x^4}\,dx$$

5



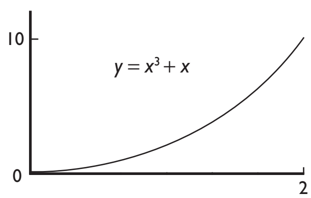






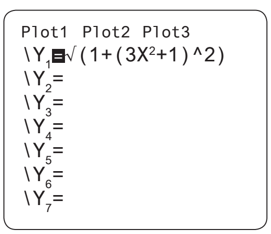
d'où $L \approx 1{,}548$

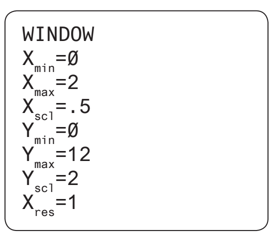
b)

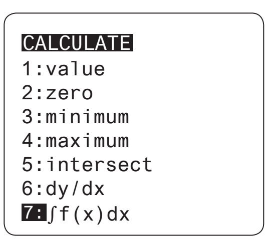


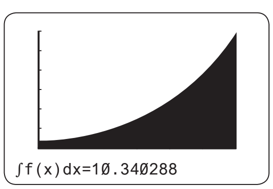
$$\frac{dy}{dx} = 3x^2 + 1$$

$$L = \int_0^2 \sqrt{1 + (3x^2 + 1)^2}\, dx$$









d'où $L \approx 10{,}340$

> with(plots):
> f:=x->x^3+x;

$$f := x \to x^3 + x$$

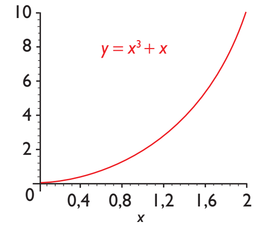
> Diff(f(x),x)=diff(f(x),x);

$$\frac{\partial}{\partial x}(x^3 + x) = 3x^2 + 1$$

> d1:=x->diff(f(x),x);

$$d1 := x \to \operatorname{diff}(f(x), x)$$

> plot(f(x),x=0..2);



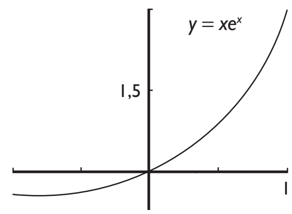
> Int((1+d1(x)^2)^(1/2),x=0..2);

$$\int_0^2 \sqrt{1 + (3x^2 + 1)^2}\, dx$$

> L:=evalf(Int((1+d1(x)^2)^(1/2),x=0..2));

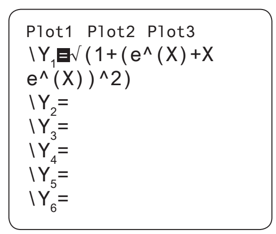
$$L := 10.34028849$$

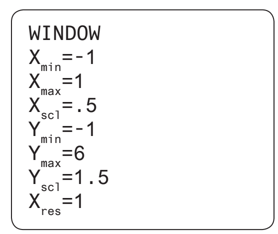
c)

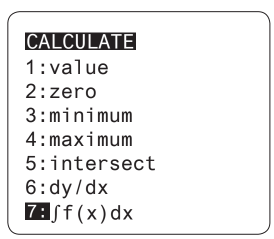


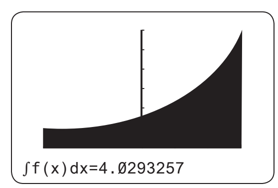
$$\frac{dy}{dx} = e^x + xe^x$$

$$L = \int_{-1}^1 \sqrt{1 + (e^x + xe^x)^2}\, dx$$









d'où $L \approx 4{,}029$

> with(plots):
> f:=x->x*exp(x);

$$f := x \to xe^x$$

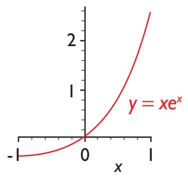
> Diff(f(x),x)=diff(f(x),x);

$$\frac{\partial}{\partial x} xe^x = e^x + xe^x$$

> d1:=x->diff(f(x),x);

$$d1 := x \to \operatorname{diff}(f(x), x);$$

> plot(f(x),x=-1..1);



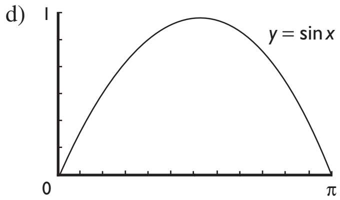
> Int((1+d1(x)^2)^(1/2),x=-1..1);

$$\int_{-1}^{1} \sqrt{1 + (e^x + xe^x)^2}\, dx$$

> L:=evalf(int((1+d1(x)^2)^(1/2),x=-1..1));

$$L := 4.029325718$$

d)



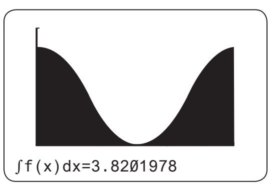
$$\frac{dy}{dx} = \cos x$$

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

```
Plot1  Plot2  Plot3
\Y1=√(1+(cos(X))^2)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
Xmin=0
Xmax=3.1415926...
Xscl=.78539816...
Ymin=1
Ymax=1.5
Yscl=.25
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



d'où $L \approx 3{,}820$

> with(plots):
> f:=x->sin(x);

$$f := \sin(x)$$

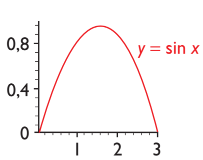
> Diff(f(x),x)=diff(f(x),x);

$$\frac{\partial}{\partial x} \sin(x) = \cos(x)$$

> d1:=x->diff(f(x),x);

$$d1 := x \to \text{diff}(f(x), x)$$

> plot(f(x),x=0..Pi);



> Int((1+d1(x)^2)^(1/2),x=0..Pi);

$$\int_0^{\pi} \sqrt{1 + \cos(x)^2}\, dx$$

> L:=evalf(int((1+d1(x)^2)^(1/2),x=0..Pi));

$$L := 3.820197788$$

e) > with(plots):
> x:=t->4*cos(2*t);

$$x := t \to 4\cos(2t)$$

> y:=t->3*sin(4*t);

$$y := t \to 3\sin(4t)$$

> Diff(x(t),t)=diff(x(t),t);

$$\frac{d}{dt}(4\cos(2t)) = -8\sin(2t)$$

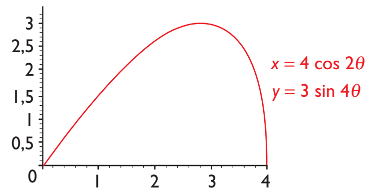
> x1:=t->diff(x(t),t);

$$x1 := t \to \frac{d}{dt} x(t)$$

> y1:=t->diff(y(t),t);

$$y1 := t \to \frac{d}{dt} y(t)$$

> plot([x(t),y(t),t=0..Pi/4]);



> Int((x1(t)^2+y1(t)^2)^(1/2),t=0..Pi/4);

$$\int_0^{\frac{\pi}{4}} \sqrt{64\sin(2t)^2 + 144\cos(4t)^2}\, dt$$

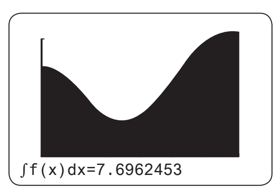
> L:=evalf(Int((x1(t)^2+y1(t)^2)^(1/2),t=0..Pi/4));
> evalf(Int((64*sin(2*x)^2+144*cos(4*x)^2)^.5,x=0..Pi/4));

$$7.696245321$$

```
Plot1  Plot2  Plot3
\Y1=√(64(sin(2X)
)^2+144(cos(4X))^2)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=.78539816...
Xscl=.39269908...
Ymin=0
Ymax=15
Yscl=5
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



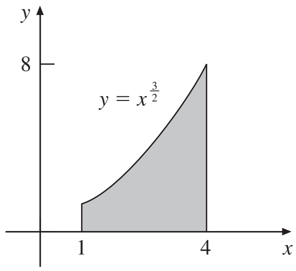
d'où $L \approx 7{,}696$

5

15. a) $A = \int_1^4 x^{\frac{3}{2}}\,dx$

$= \left.\dfrac{2x^{\frac{5}{2}}}{5}\right|_1^4$

$= \dfrac{62}{5}\text{ u}^2$



$P = 1 + 3 + 8 + L$, où L est la longueur de la courbe du point (1, 1) au point (4, 8).

$L = \int_1^4 \sqrt{1 + \left(\dfrac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx$

$= \int_1^4 \sqrt{1 + \dfrac{9x}{4}}\,dx$

$= \left.\dfrac{8}{27}\left(1 + \dfrac{9x}{4}\right)^{\frac{3}{2}}\right|_1^4$

$\approx 7{,}63$

d'où $P \approx 12 + 7{,}63$

$\approx 19{,}63$ u

b) i) $V_{0X} = \int_1^4 \pi\left(x^{\frac{3}{2}}\right)^2 dx$

$= \left.\dfrac{\pi x^4}{4}\right|_1^4$

$= \dfrac{255\pi}{4}\text{ u}^3$

ii) $V_{0Y} = \int_1^4 2\pi xy\,dx$

$= 2\pi\int_1^4 x^{\frac{5}{2}}\,dx \qquad \left(y = x^{\frac{3}{2}}\right)$

$= \left.\dfrac{4\pi}{7}x^{\frac{7}{2}}\right|_1^4$

$= \dfrac{508\pi}{7}\text{ u}^3$

c) i) Volume perpendiculaire à l'axe des x.

$V = \int_1^4 y^2\,dx$

$= \int_1^4 x^3\,dx \qquad \left(y = x^{\frac{3}{2}}\right)$

$= \left.\dfrac{x^4}{4}\right|_1^4$

$= \dfrac{255}{4}\text{ u}^3$

ii) Volume perpendiculaire à l'axe des y.

$V_T = V_1 + V_2$, où

$V_1 = \int_0^1 (3)(3)\,dx$

$= 9x\big|_0^1$

$= 9$

$V_2 = \int_1^8 (4 - x)^2\,dy$

$= \int_1^8 \left(4 - y^{\frac{2}{3}}\right)^2 dy \qquad \left(x = y^{\frac{2}{3}}\right)$

$= \int_1^8 \left(16 - 8y^{\frac{2}{3}} + y^{\frac{4}{3}}\right) dy$

$= \left.\left(16y - \dfrac{24}{5}y^{\frac{5}{3}} + \dfrac{3}{7}y^{\frac{7}{3}}\right)\right|_1^8$

$= \left(128 - \dfrac{24}{5}(32) + \dfrac{3}{7}(128)\right) - \left(16 - \dfrac{24}{5} + \dfrac{3}{7}\right)$

$= \dfrac{617}{35}$

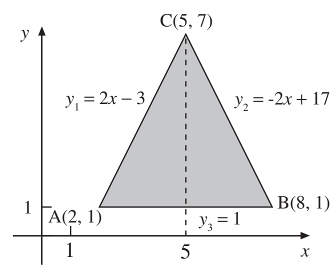
d'où $V_T = 9 + \dfrac{617}{35} = \dfrac{932}{35}\text{ u}^3$

16. a) Déterminons d'abord l'équation des droites passant par les sommets du triangle.

$y_1 = 2x - 3$ passe par A(2, 1) et C(5, 7).

$y_2 = -2x + 17$ passe par C(5, 7) et B(8, 1).

$y_3 = 1$ passe par A(2, 1) et B(8, 1).



i) Autour de l'axe des x.

$V_{0X} = 2\left[\pi\int_2^5 (2x - 3)^2\,dx\right] - 6\pi$

$= 2\pi\left.\dfrac{(2x - 3)^3}{6}\right|_2^5 - 6\pi$

$= \dfrac{342\pi}{3} - 6\pi$

$= 108\pi\text{ u}^3$

ii) Autour de l'axe des y.

$V_{0Y} = 2\pi\int_2^5 x(y_1 - 1)\,dx + 2\pi\int_5^8 x(y_2 - 1)\,dx$

$= 2\pi\int_2^5 x(2x - 4)\,dx + 2\pi\int_5^8 x(-2x + 16)\,dx$

$= 2\pi\left.\left(\dfrac{2x^3}{3} - 2x^2\right)\right|_2^5 + 2\pi\left.\left(\dfrac{-2x^3}{3} + 8x^2\right)\right|_5^8$

$= 2\pi\left(\dfrac{108}{3}\right) + 2\pi\left(\dfrac{162}{3}\right)$

$= 180\pi\text{ u}^3$

b) i) Autour de $x = 5$,

nous obtenons un cône de rayon 3 et de hauteur 6,

d'où $V = \dfrac{\pi(3)^2 6}{3} = 18\pi\text{ u}^3$.

ii) Autour de $y = 1$,

nous obtenons deux cônes de rayon 6 et de hauteur 3,

d'où $V = 2\left[\dfrac{\pi(6)^2 3}{3}\right] = 72\pi\text{ u}^3$.

17. a) En posant $f(t) = 1000$ et $i = 0{,}1$, nous obtenons

$$P = \int_0^{+\infty} 1000e^{-0{,}1t}\,dt$$
$$= \lim_{M\to+\infty} \int_0^M 1000e^{-0{,}1t}\,dt$$
$$= \lim_{M\to+\infty} \left[\frac{-1000e^{-0{,}1t}}{0{,}1}\right]\Bigg|_0^M$$
$$= \lim_{M\to+\infty} \left[\frac{-1000e^{-0{,}1M}}{0{,}1} + \frac{1000}{0{,}1}\right]$$
$$= 0 + 10\,000$$

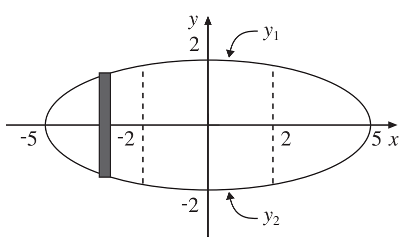
d'où $P = 10\,000$ \$.

b) En posant $f(t) = 1000(1{,}06)^t$ et $i = 0{,}1$, nous obtenons

$$P = \int_0^{+\infty} 1000(1{,}06)^t e^{-0{,}1t}\,dt$$
$$= \lim_{M\to+\infty} \int_0^M 1000(1{,}06e^{-0{,}1})^t\,dt$$
$$= \lim_{M\to+\infty} \left[\frac{1000(1{,}06e^{-0{,}1})^t}{\ln(1{,}06e^{-0{,}1})}\right]\Bigg|_0^M$$
$$= \lim_{M\to+\infty} \left[\frac{1000(1{,}06e^{-0{,}1})^M}{\ln(1{,}06e^{-0{,}1})} - \frac{1000}{\ln(1{,}06e^{-0{,}1})}\right]$$
$$= 0 + 23\,962{,}949\ldots$$

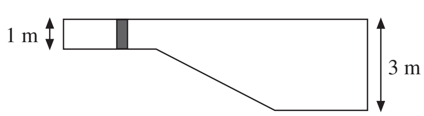
d'où $P \approx 23\,962{,}95$ \$.

18. Déterminons d'abord l'équation de l'ellipse.



$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

$$y_1 = \frac{2}{5}\sqrt{25 - x^2} \quad \text{et} \quad y_2 = \frac{-2}{5}\sqrt{25 - x^2}$$



Le volume de chaque section perpendiculaire à l'axe des x est donné par

$$\Delta V = p(y_1 - y_2)\,\Delta x = p\left(\frac{4}{5}\sqrt{25 - x^2}\right)\Delta x$$

où p correspond à la profondeur de la piscine. Calculons le volume de la piscine.

Sur [-5, -2], nous avons $p = 1$, ainsi

$$V_1 = \frac{4}{5}\int_{-5}^{-2} \sqrt{25 - x^2}\,dx$$
$$= \frac{4}{5}\left[\frac{x\sqrt{25 - x^2}}{2} + \frac{25}{2}\,\text{Arc}\sin\left(\frac{x}{5}\right)\right]\Bigg|_{-5}^{-2}$$

(formule 3, page 472)

$$= \frac{4}{5}\left[-9{,}726\,53\ldots + \frac{25\pi}{4}\right]$$
$$= 7{,}926\,73\ldots$$

Sur [2, 5], nous avons $p = 3$, ainsi

$V_2 = 3V_1 = 23{,}780\,20\ldots$

Sur [-2, 2], la profondeur de la piscine varie de 1 m à 3 m, cependant à cause de la symétrie de la piscine, nous pouvons poser $p = 2$, ainsi

$$V_2 = 2\left(\frac{4}{5}\right)\int_{-2}^{2} \sqrt{25 - x^2}\,dx$$
$$= \frac{16}{5}\int_0^2 \sqrt{25 - x^2}\,dx$$
$$= \frac{16}{5}\left[\frac{x\sqrt{25 - x^2}}{2} + \frac{25}{2}\,\text{Arc}\sin\left(\frac{x}{5}\right)\right]\Bigg|_0^2$$
$$= \frac{16}{5}[9{,}726\,53\ldots - 0]$$
$$= 31{,}124\,91\ldots$$

Ainsi, le volume V total de la piscine est

$V = V_1 + V_2 + V_3 \approx 62{,}832$

d'où la capacité maximale est d'environ 62 832 litres.

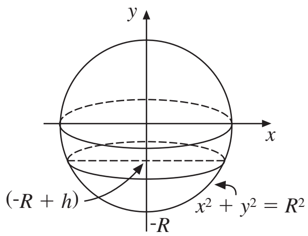
19.
$$dv = \int \frac{-20}{t^3}\,dt$$
$$v = \frac{10}{t^2} + C$$

Puisque à $t = 1$, $v = 10$

alors $10 = 10 + C$, donc $C = 0$

ainsi
$$v = \frac{10}{t^2}$$
$$s = \int_1^{+\infty} \frac{10}{t^2}\,dt$$
$$= \lim_{M\to+\infty} \int_1^M \frac{10}{t^2}\,dt$$
$$= \lim_{M\to+\infty} \left(\frac{-10}{t}\right)\Bigg|_1^M$$
$$= \lim_{M\to+\infty} \left(\frac{-10}{M} + \frac{10}{1}\right)$$
$$= 10 \text{ mètres}$$

20. a)



$$V_{0Y} = \pi\int_{-R}^{-R+h} x^2\,dy$$
$$= \pi\int_{-R}^{-R+h} (R^2 - y^2)\,dy$$
$$= \pi\left(R^2 y - \frac{y^3}{3}\right)\Bigg|_{-R}^{-R+h}$$
$$= \pi\left[\left(R^2(-R + h) - \frac{(-R + h)^3}{3}\right) - \left(-R^3 + \frac{R^3}{3}\right)\right]$$

d'où $V = \pi h^2\left(R - \dfrac{h}{3}\right)$ (exprimé en m³)

b) En posant $R = 10$, nous obtenons $V = \pi h^2\left(10 - \dfrac{h}{3}\right)$

i) Déterminons le volume lorsque $h = 5$.

$V = \pi(5)^2\left(10 - \dfrac{5}{3}\right) = \dfrac{625\pi}{3}$

Puisque le rythme est constant, $(0{,}05)t = \dfrac{625\pi}{3}$

d'où $t = 13\,089{,}96\ldots$ s, c'est-à-dire environ 3,64 heures.

ii) $(0{,}05)t = \dfrac{2000\pi}{3}$

d'où $t = 41\,887{,}90\ldots$ s, c'est-à-dire environ 11,64 heures.

c) Nous cherchons $\dfrac{dh}{dt}$.

$\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt}$ (notation de Leibniz)

$$0{,}05 = \frac{d}{dh}\left[\pi h^2\left(10 - \frac{h}{3}\right)\right]\frac{dh}{dt}$$

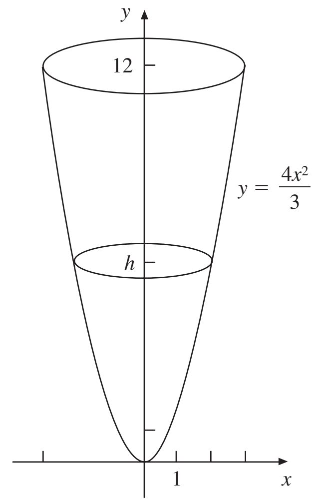
$$0{,}05 = (20\pi h - \pi h^2)\frac{dh}{dt}$$

Ainsi $\dfrac{dh}{dt} = \dfrac{0{,}05}{\pi h(20 - h)}$

i) d'où $\dfrac{dh}{dt}\Big|_{h=1} = \dfrac{0{,}05}{19\pi} \approx 0{,}000\,84$ m/s

ii) d'où $\dfrac{dh}{dt}\Big|_{h=9} = \dfrac{0{,}05}{99\pi} \approx 0{,}000\,16$ m/s

21.



a) $V = \pi\displaystyle\int_0^{12} x^2\,dy$

$$= \pi\int_0^{12} \frac{3y}{4}\,dy$$
$$= \frac{3\pi}{4}\left[\frac{y^2}{2}\right]\Bigg|_0^{12}$$
$$= \frac{3\pi}{4}[72 - 0]$$

d'où $V = 54\pi$ cm³

b) Puisque $\dfrac{dV}{dt} = -3$

$dV = -3\,dt$

$\int dV = -\int 3\,dt$

$V = -3t + C$

En posant $t = 0$ et $V = 54\pi$, nous obtenons $C = 54\pi$
d'où $V(t) = -3t + 54\pi$.

$$V(h) = \pi\int_0^{h} \frac{3y}{4}\,dy$$
$$= \frac{3\pi}{4}\left[\frac{y^2}{2}\right]\Bigg|_0^{h}$$
$$= \frac{3\pi}{4}\left[\frac{h^2}{2} - 0\right]$$

d'où $V(h) = \dfrac{3\pi h^2}{8}$ (en cm³)

c) $\dfrac{dV}{dt} = \dfrac{dV}{dt}\dfrac{dh}{dt}$ (notation de Leibniz)

$$-3 = \frac{d}{dh}\left[\frac{3\pi h^2}{8}\right]\frac{dh}{dt}$$

$$-3 = \frac{3\pi h}{4}\frac{dh}{dt}$$

d'où $\dfrac{dh}{dt} = \dfrac{-4}{\pi h}$ (exprimée en cm/s)

d) i) $\dfrac{dh}{dt}\Big|_{h=6} = \dfrac{-4}{\pi 6} \approx -0{,}21$ cm/s

ii) Lorsque $V(h) = \dfrac{54\pi}{2}$

$\dfrac{3\pi h^2}{8} = 27\pi$, ainsi $h = \sqrt{72}$

d'où $\left.\frac{dh}{dt}\right|_{V=27\pi} = \frac{-4}{\pi\sqrt{72}} \approx -0{,}15$ cm/s

iii) Lorsque $t = 50$, $V = -3(50) + 54\pi = 54\pi - 150$

En posant $V(h) = 54\pi - 150$, nous obtenons

$$\frac{3\pi h^2}{8} = 54\pi - 150, \text{ ainsi } h = 4{,}083\ldots$$

d'où $\left.\frac{dh}{dt}\right|_{t=50} = \frac{-4}{\pi(4{,}083\ldots)} \approx -0{,}31$ cm/s

e) En posant $V(t) = 0$, nous obtenons

$-3t + 54\pi = 0$,

d'où $t \approx 56{,}55$ s

22. a) Évaluons d'abord $\int_{-\infty}^{+\infty} f(x)\,dx$.

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{+\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{+\infty} cxe^{-3x}\,dx$$

(définition de $f(x)$)

$$= 0 + \lim_{M\to+\infty} c\int_0^M xe^{-3x}\,dx$$

$$= c \lim_{M\to+\infty}\left[\frac{xe^{-3x}}{-3} - \frac{e^{-3x}}{(-3)^2}\right]\Bigg|_0^M$$

(formule 38, page 474)

$$= c \lim_{M\to+\infty}\left[\left(\frac{M}{-3e^{3M}} - \frac{1}{9e^{3M}}\right) - \left(0 - \frac{1}{9}\right)\right]$$

$$= c\left(\frac{1}{9}\right)$$

En posant $\int_{-\infty}^{+\infty} f(x)\,dx = 1$, nous obtenons

$$c\left(\frac{1}{9}\right) = 1$$

d'où $c = 9$

b) i) $\int_0^2 f(x)\,dx = \int_0^2 9xe^{-3x}\,dx$

$$= 9\left[\frac{xe^{-3x}}{-3} - \frac{e^{-3x}}{9}\right]\Bigg|_0^2 \quad \text{(formule 38, page 474)}$$

$$= 9\left[\left(\frac{2e^{-6}}{-3} - \frac{e^{-6}}{9}\right) - \left(0 - \frac{1}{9}\right)\right]$$

$$= \left(1 - \frac{7}{e^6}\right)$$

ii) Puisque $\int_0^{+\infty} f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^{+\infty} f(x)\,dx$

alors $1 = \left(1 - \frac{7}{e^6}\right) + \int_2^{+\infty} f(x)\,dx$

d'où $\int_2^{+\infty} f(x)\,dx = \frac{7}{e^6}$

c) $\int_{-\infty}^{+\infty} x\,f(x)\,dx = \int_{-\infty}^{0} x\,f(x)\,dx + \int_{0}^{+\infty} x\,f(x)\,dx$

$$= 0 + \int_0^{+\infty} x(9xe^{-3x})\,dx$$

$$= 9\lim_{M\to+\infty}\int_0^M x^2e^{-3x}\,dx$$

$$= 9\lim_{M\to+\infty}\left[\frac{x^2e^{-3x}}{-3} - \frac{2xe^{-3x}}{(-3)^2} + \frac{2e^{-3x}}{(-3)^3}\right]\Bigg|_0^M$$

(formule 39, page 474)

$$= 9\lim_{M\to+\infty}\left[\left(\frac{M^2}{-3e^{3M}} - \frac{2M}{9e^{3M}} - \frac{2}{27e^{3M}}\right) - \left(\frac{-2}{27}\right)\right]$$

$$= 9\left(\frac{2}{27}\right) = \frac{2}{3}$$

23. $\int_0^{+\infty} x^ne^{-x}dx = \lim_{M\to+\infty}\int_0^M x^ne^{-x}\,dx$

$$= \lim_{M\to+\infty}\left[\left(\frac{x^ne^{-x}}{-1}\right)\Bigg|_0^M - \frac{n}{-1}\int_0^M x^{n-1}e^{-x}\,dx\right]$$

(formule 56, page 474)

$$= \lim_{M\to+\infty}\left(\frac{-x^n}{e^x}\right)\Bigg|_0^M + \lim_{M\to+\infty}\left(n\int_0^M x^{n-1}e^{-x}\,dx\right)$$

$$= \lim_{M\to+\infty}\left[\frac{-M^n}{e^M} + 0\right] + n\lim_{M\to+\infty}\int_0^M x^{n-1}e^{-x}\,dx$$

$$= 0 + n\int_0^{+\infty} x^{n-1}e^{-x}\,dx$$

Ainsi $\int_0^{+\infty} x^ne^{-x}\,dx = n\left[\int_0^{+\infty} x^{n-1}e^{-x}\,dx\right]$

$$= n(n-1)\left[\int_0^{+\infty} x^{n-2}e^{-x}\,dx\right]$$

(de façon analogue)

$$= n(n-1)(n-2)\left[\int_0^{+\infty} x^{n-3}e^{-x}\,dx\right]$$

$$= n(n-1)(n-2)\ldots(3)(2)(1)\left[\int_0^{+\infty} e^{-x}\,dx\right]$$

$$= n!\left[\lim_{M\to+\infty}\int_0^M e^{-x}\,dx\right]$$

$$= n!\left[\lim_{M\to+\infty}(-e^{-x})\Big|_0^M\right]$$

$$= n!\left[\lim_{M\to+\infty}(-e^{-M} + 1)\right]$$

$$= n![1]$$

$$= n!$$

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