Solutionnaire
Exercices récapitulatifs

Chapitre 4 (page 254)

1. a) $I = \int 5t \cos t \, dt$

$u = 5t$ $du = 5\,dt$	$dv = \cos t \, dt$ $v = \sin t$

$$I = 5t \sin t - 5\int \sin t \, dt$$
$$= 5t \sin t + 5 \cos t + C$$
$$= 5(t \sin t + \cos t) + C$$

b) $I = \int x^2 e^{\frac{-x}{3}} dx$

$u = x^2$ $du = 2x\,dx$	$dv = e^{\frac{-x}{3}} dx$ $v = -3e^{\frac{-x}{3}}$

$$I = -3x^2 e^{\frac{-x}{3}} + 6\int x e^{\frac{-x}{3}} dx$$

$u = x$ $du = dx$	$dv = e^{\frac{-x}{3}} dx$ $v = -3e^{\frac{-x}{3}}$

$$I = -3x^2 e^{\frac{-x}{3}} + 6\left[-3xe^{\frac{-x}{3}} + 3\int e^{\frac{-x}{3}} dx\right]$$
$$= -3x^2 e^{\frac{-x}{3}} - 18xe^{\frac{-x}{3}} - 54e^{\frac{-x}{3}} + C$$

c) $I = \int x \operatorname{Arc\,sec} x \, dx$

$u = \operatorname{Arc\,sec} x$ $du = \dfrac{1}{x\sqrt{x^2-1}} dx$	$dv = x\,dx$ $v = \dfrac{x^2}{2}$

$$I = \frac{x^2 \operatorname{Arc\,sec} x}{2} - \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}} dx$$
$$= \frac{x^2 \operatorname{Arc\,sec} x}{2} - \frac{1}{4}\int \frac{1}{u^{\frac{1}{2}}} du \qquad (u = x^2 - 1)$$
$$= \frac{x^2 \operatorname{Arc\,sec} x}{2} - \frac{1}{2}u^{\frac{1}{2}} + C$$
$$= \frac{x^2 \operatorname{Arc\,sec} x}{2} - \frac{\sqrt{x^2-1}}{2} + C$$

d) $I = \int e^{-x} \cos x \, dx$

$u = e^{-x}$ $du = -e^{-x} dx$	$dv = \cos x \, dx$ $v = \sin x$

$$I = e^{-x} \sin x + \int e^{-x} \sin dx$$

$u = e^{-x}$ $du = -e^{-x} dx$	$dv = \sin x \, dx$ $v = -\cos x$

$$I = e^{-x} \sin x + \left[-e^{-x} \cos x - \int e^{-x} \cos x \, dx\right]$$
$$2I = e^{-x} \sin x - e^{-x} \cos x + C_1$$
$$I = \frac{e^{-x} \sin x - e^{-x} \cos x}{2} + C$$

e) $I = \int \dfrac{\ln^2 y}{y^2} dy$

$u = \ln^2 y$ $du = 2 \ln y \left(\dfrac{1}{y}\right) dy$	$dv = y^{-2} dy$ $v = \dfrac{y^{-1}}{-1}$

$$I = \frac{-1}{y}\ln^2 y + 2\int \frac{\ln y}{y^2} dy$$

$u = \ln y$ $du = \dfrac{1}{y} dy$	$dv = y^{-2} dy$ $v = \dfrac{y^{-1}}{-1}$

$$I = \frac{-1}{y}\ln^2 y + 2\left[\frac{-\ln y}{y} + \int y^{-2} dy\right]$$
$$= \frac{-1}{y}\ln^2 y - \frac{2 \ln y}{y} - 2\left(\frac{1}{y}\right) + C$$
$$= \frac{-1}{y}(\ln^2 y + 2 \ln y + 2) + C$$

f) $I = \int xe^x \cos x \, dx$

$u = x$ $du = dx$	$dv = e^x \cos x \, dx$ $v = \int e^x \cos x \, dx$

Calculons $I_1 = \int e^x \cos x \, dx$

$u = e^x$ $du = e^x dx$	$dv = \cos x \, dx$ $v = \sin x$

$$I_1 = e^x \sin x - \int e^x \sin x \, dx$$

$u = e^x$ $du = e^x dx$	$dv = \sin x \, dx$ $v = -\cos x$

$$I_1 = e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx\right]$$
$$I_1 = e^x \sin x + e^x \cos x - I_1$$
$$2I_1 = e^x \sin x + e^x \cos x + C_1$$
$$I_1 = \frac{e^x \sin x + e^x \cos x}{2} + C_2$$

donc

$$I = x\left(\frac{e^x \sin x + e^x \cos x}{2}\right) - \frac{1}{2}\int (e^x \sin x + e^x \cos x)\,dx$$

$$= x\left(\frac{e^x \sin x + e^x \cos x}{2}\right) - \frac{1}{2}\left[\frac{e^x \sin x - e^x \cos x}{2} + \frac{e^x \sin x + e^x \cos x}{2}\right] + C$$

$$I = \frac{xe^x \sin x + xe^x \cos x - e^x \sin x}{2} + C$$

g) $I = \int x \sin x\,dx$

$u = x$ $du = dx$	$dv = \sin x\,dx$ $v = -\cos x$

$$I = \int x \sin x\,dx$$
$$= -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + C$$

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \sin x\,dx = (-x\cos x + \sin x)\Big|_{\frac{-\pi}{2}}^{\frac{\pi}{2}}$$
$$= \left(\frac{-\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\cos\left(\frac{-\pi}{2}\right) + \sin\left(\frac{-\pi}{2}\right)\right)$$
$$= 2$$

2. a) $I = \int (16 + 8\cos x + \cos^2 x)\,dx$

$$= 16x + 8\sin x + \int \cos^2 x\,dx$$
$$= 16x + 8\sin x + \int \frac{1 + \cos 2x}{2}\,dx$$
$$= 16x + 8\sin x + \frac{x}{2} + \frac{\sin 2x}{4} + C$$
$$= \frac{33x}{2} + 8\sin x + \frac{\sin 2x}{4} + C$$

b) $I = \int \frac{\sec^2\theta}{\tan\theta}\,d\theta + \int \frac{\cos\theta}{\sin\theta}\,\frac{1}{\sin\theta}\,d\theta$

$u = \tan\theta$ $du = \sec^2\theta\,d\theta$

$$I = \int \frac{1}{u}\,du + \int \csc\theta\cot\theta\,d\theta$$
$$= \ln|u| + (-\csc\theta) + C$$
$$= \ln|\tan\theta| - \csc\theta + C$$

c) $I = \int (1 - 2\sec^2 x + \sec^4 x)\,dx$

$$= x - 2\tan x + \int \sec^2 x \sec^2 x\,dx$$
$$= x - 2\tan x + \int (1 + \tan^2 x)\sec^2 x\,dx$$
$$= x - 2\tan x + \int \sec^2 x\,dx + \int \tan^2 x \sec^2 x\,dx$$
$$= x - 2\tan x + \tan x + \frac{\tan^3 x}{3} + C \quad (u = \tan x)$$
$$= x - \tan x + \frac{\tan^3 x}{3} + C$$

d) $I = \int (\sin^2 2t)^3\,dt$

$$= \int \left(\frac{1 - \cos 4t}{2}\right)^3 dt$$
$$= \frac{1}{8}\int (1 - 3\cos 4t + 3\cos^2 4t - \cos^3 4t)\,dt$$
$$= \frac{1}{8}\left[t - \frac{3\sin 4t}{4} + 3\int \frac{1 + \cos 8t}{2}\,dt - \int \cos 4t\,(1 - \sin^2 4t)\,dt\right]$$
$$= \frac{1}{8}\left[t - \frac{3\sin 4t}{4} + \frac{3}{2}t + \frac{3}{2}\,\frac{\sin 8t}{8} - \int (\cos 4t - \cos 4t \sin^2 4t)\,dt\right]$$
$$= \frac{1}{8}\left[t - \frac{3\sin 4t}{4} + \frac{3}{2}t + \frac{3\sin 8t}{16} - \frac{\sin 4t}{4} + \frac{\sin^3 4t}{12}\right] + C$$
$$= \frac{1}{8}\left[\frac{5t}{2} - \sin 4t + \frac{3\sin 8t}{16} + \frac{\sin^3 4t}{12}\right] + C$$

e) $I = \int \sec^3 x\,(\tan^2 x)^2\,dx$

$$= \int \sec^3 x\,(\sec^2 x - 1)^2\,dx$$
$$= \int \sec^7 x\,dx - 2\int \sec^5 x\,dx + \int \sec^3 x\,dx$$
$$= \left[\frac{\sec^5 x \tan x}{6} + \frac{5}{6}\int \sec^5 x\,dx\right] - 2\int \sec^5 x\,dx + \int \sec^3 x\,dx$$
$$= \frac{\sec^5 x \tan x}{6} - \frac{7}{6}\int \sec^5 x\,dx + \int \sec^3 x\,dx$$
$$= \frac{\sec^5 x \tan x}{6} - \frac{7}{6}\left[\frac{\sec^3 x \tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx\right] + \int \sec^3 x\,dx$$
$$= \frac{\sec^5 x \tan x}{6} - \frac{7\sec^3 x \tan x}{24} + \frac{1}{8}\int \sec^3 x\,dx$$
$$= \frac{\sec^5 x \tan x}{6} - \frac{7\sec^3 x \tan x}{24} + \frac{1}{8}\left[\frac{\sec x \tan x + \ln|\sec x + \tan x|}{2}\right] + C$$
$$= \frac{\sec^5 x \tan x}{6} - \frac{7\sec^3 x \tan x}{24} + \frac{\sec x \tan x}{16} + \frac{\ln|\sec x + \tan x|}{16} + C$$

f) $I = \int \left(\frac{\cos x + \cos 5x}{2}\right)^2 dx$

$$= \frac{1}{4}\int (\cos^2 x + 2\cos 5x \cos x + \cos^2 5x)\,dx$$
$$= \frac{1}{4}\int \left(\frac{1 + \cos 2x}{2} + 2\left(\frac{\cos 4x + \cos 6x}{2}\right) + \frac{1 + \cos 10x}{2}\right)dx$$
$$= \frac{1}{4}\left[\frac{x}{2} + \frac{\sin 2x}{4} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \frac{x}{2} + \frac{\sin 10x}{20}\right] + C$$
$$= \frac{1}{4}\left(x + \frac{\sin 2x}{4} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \frac{\sin 10x}{20}\right) + C$$

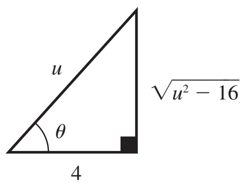
g) $I = \int \tan^2 x \sec^2 x \sec^2 x\,dx$

$$= \int \tan^2 x\,(1 + \tan^2 x)\sec^2 x\,dx$$
$$= \int (\tan^2 x \sec^2 x + \tan^4 x \sec^2 x)\,dx$$
$$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^4 x\, dx = \left(\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5}\right)\Bigg|_0^{\frac{\pi}{4}}$$
$$= \left(\frac{1}{3}\tan^3\frac{\pi}{4} + \frac{1}{5}\tan^5\frac{\pi}{4}\right) - 0$$
$$= \frac{1}{3} + \frac{1}{5}$$
$$= \frac{8}{15}$$

3. a) $I = \int \frac{\sqrt{u^2-16}}{3u}\, du$

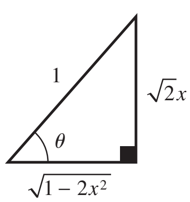
$$u^2 = 16\sec^2\theta$$
$$u = 4\sec\theta$$
$$du = 4\sec\theta\tan\theta\, d\theta$$
$$\theta = \text{Arc sec}\left(\frac{u}{4}\right)$$



$$I = \frac{1}{3}\int \frac{\sqrt{16\sec^2\theta - 16}}{4\sec\theta}\, 4\sec\theta\tan\theta\, d\theta$$
$$= \frac{1}{3}\int 4\tan^2\theta\, d\theta$$
$$= \frac{4}{3}\int(\sec^2\theta - 1)\, d\theta$$
$$= \frac{4}{3}(\tan\theta - \theta) + C$$
$$= \frac{4}{3}\left(\frac{\sqrt{u^2-16}}{4} - \text{Arc sec}\,\frac{u}{4}\right) + C$$
$$= \frac{\sqrt{u^2-16}}{3} - \frac{4}{3}\,\text{Arc sec}\,\frac{u}{4} + C$$

b) $I = \int \frac{1}{(1-2x^2)^{\frac{5}{2}}}\, dx$

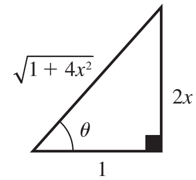
$$2x^2 = \sin^2\theta$$
$$x = \frac{1}{\sqrt{2}}\sin\theta$$
$$dx = \frac{1}{\sqrt{2}}\cos\theta\, d\theta$$
$$\theta = \text{Arc sin}\left(\sqrt{2}x\right)$$



$$I = \int \frac{1}{(1-\sin^2\theta)^{\frac{5}{2}}}\,\frac{1}{\sqrt{2}}\cos\theta\, d\theta$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{\cos^5\theta}\cos\theta\, d\theta$$
$$= \frac{1}{\sqrt{2}}\int \sec^4\theta\, d\theta$$
$$= \frac{1}{\sqrt{2}}\int \sec^2\theta(1+\tan^2\theta)\, d\theta$$
$$= \frac{1}{\sqrt{2}}\int(\sec^2\theta + \sec^2\theta\tan^2\theta)\, d\theta$$
$$= \frac{1}{\sqrt{2}}\left(\tan\theta + \frac{\tan^3\theta}{3}\right) + C$$
$$= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}x}{\sqrt{1-2x^2}} + \frac{1}{3}\left(\frac{\sqrt{2}x}{\sqrt{1-2x^2}}\right)^3\right) + C$$
$$= \frac{x}{\sqrt{1-2x^2}} + \frac{2x^3}{3\sqrt{(1-2x^2)^3}} + C$$

c) $I = \int \frac{4}{(1+4x^2)^2}\, dx$

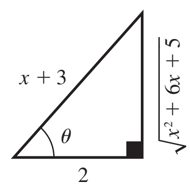
$$4x^2 = \tan^2\theta$$
$$2x = \tan\theta$$
$$dx = \frac{1}{2}\sec^2\theta\, d\theta$$
$$\theta = \text{Arc tan}\,(2x)$$



$$I = 4\int \frac{1}{(1+\tan^2\theta)^2}\,\frac{1}{2}\sec^2\theta\, d\theta$$
$$= 2\int \frac{1}{\sec^4\theta}\sec^2\theta\, d\theta$$
$$= 2\int \cos^2\theta\, d\theta$$
$$= 2\int \frac{1+\cos 2\theta}{2}\, d\theta$$
$$= \theta + \frac{\sin 2\theta}{2} + C$$
$$= \text{Arc tan}\,2x + \frac{2\sin\theta\cos\theta}{2} + C$$
$$= \text{Arc tan}\,2x + \frac{2x}{1+4x^2} + C$$

d) $I = \int \sqrt{x^2+6x+5}\, dx = \int \sqrt{(x+3)^2 - 4}\, dx$

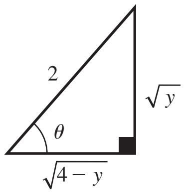
$$(x+3)^2 = 4\sec^2\theta$$
$$x+3 = 2\sec\theta$$
$$x = 2\sec\theta - 3$$
$$dx = 2\sec\theta\tan\theta\, d\theta$$
$$\theta = \text{Arc sec}\left(\frac{x+3}{2}\right)$$



$$I = \int \sqrt{4\sec^2\theta - 4}\; 2\sec\theta\tan\theta\, d\theta$$
$$= 2\cdot 2\int \tan\theta\sec\theta\tan\theta\, d\theta$$
$$= 4\int \tan^2\theta\sec\theta\, d\theta$$
$$= 4\int(\sec^2\theta - 1)\sec\theta\, d\theta$$
$$= 4\int(\sec^3\theta - \sec\theta)\, d\theta$$
$$= 4\left[\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} - \ln|\sec\theta + \tan\theta|\right] + C$$
$$= 2\sec\theta\tan\theta - 2\ln|\sec\theta + \tan\theta| + C$$
$$= 2\left(\frac{x+3}{2}\right)\frac{\sqrt{x^2+6x+5}}{2} - 2\ln\left|\frac{x+3}{2} + \frac{\sqrt{x^2+6x+5}}{2}\right| + C$$
$$= \frac{(x+3)\sqrt{x^2+6x+5}}{2} - 2\ln\left|\frac{x+3+\sqrt{x^2+6x+5}}{2}\right| + C$$

e) $I = \displaystyle\int \frac{1}{y\sqrt{1-\frac{y}{4}}}\,dy$

$$\frac{y}{4} = \sin^2\theta$$
$$y = 4\sin^2\theta$$
$$dy = 8\sin\theta\cos\theta\,d\theta$$
$$\theta = \text{Arc}\sin\left(\frac{\sqrt{y}}{2}\right)$$



$$I = \int \frac{1}{4\sin^2\theta\sqrt{1-\sin^2\theta}}\,8\sin\theta\cos\theta\,d\theta$$
$$= 2\int \csc\theta\,d\theta$$
$$= 2\ln\left|\csc\theta - \cot\theta\right| + C$$
$$= 2\ln\left|\frac{2}{\sqrt{y}} - \frac{\sqrt{4-y}}{\sqrt{y}}\right| + C$$
$$= 2\ln\left|\frac{2-\sqrt{4-y}}{\sqrt{y}}\right| + C$$

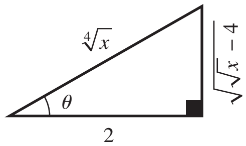
f) $I = \displaystyle\int \frac{2\sin\theta - 3\cos\theta}{1+\cos\theta}\,d\theta$

$$\sin\theta = \frac{2u}{1+u^2} \qquad d\theta = \frac{2}{1+u^2}\,du$$
$$\cos\theta = \frac{1-u^2}{1+u^2} \qquad u = \tan\left(\frac{\theta}{2}\right)$$

$$I = \int \frac{2\left(\frac{2u}{1+u^2}\right) - 3\left(\frac{1-u^2}{1+u^2}\right)}{1+\frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du$$
$$= \int \frac{4u-3+3u^2}{1+u^2}\,du$$
$$= \int \frac{4u}{1+u^2}\,du - 3\int \frac{1}{1+u^2}\,du + 3\int \frac{u^2}{1+u^2}\,du$$
$$= \frac{4\ln\left|1+u^2\right|}{2} - 3\,\text{Arc}\tan u + 3\int\left(1-\frac{1}{u^2+1}\right)du + C$$
$$= 2\ln\left|1+u^2\right| - 3\,\text{Arc}\tan u + 3u - 3\,\text{Arc}\tan u + C$$
$$= 2\ln\left|1+\tan^2\left(\frac{\theta}{2}\right)\right| + 3\tan\left(\frac{\theta}{2}\right) - 6\,\text{Arc}\tan\left(\tan\left(\frac{\theta}{2}\right)\right) + C$$
$$= 2\ln\left|1+\tan^2\left(\frac{\theta}{2}\right)\right| + 3\tan\left(\frac{\theta}{2}\right) - 6\left(\frac{\theta}{2}\right) + C$$
$$= 2\ln\left(1+\tan^2\left(\frac{\theta}{2}\right)\right) + 3\tan\left(\frac{\theta}{2}\right) - 3\theta + C$$

g) $I = \displaystyle\int \frac{1}{x\sqrt{\sqrt{x}-4}}\,dx$

$$\sqrt[4]{x} = 2\sec\theta$$
$$x = 16\sec^4\theta$$
$$dx = 64\sec^4\theta\tan\theta\,d\theta$$
$$\theta = \text{Arc}\sec\left(\frac{\sqrt[4]{x}}{2}\right)$$



$$I = \int \frac{64\sec^4\theta\tan\theta}{16\sec^4\theta\sqrt{4\sec^2\theta - 4}}\,d\theta$$
$$= 4\int \frac{\tan\theta}{2\sqrt{\tan^2\theta}}\,d\theta$$
$$= 2\int d\theta$$
$$= 2\theta + C$$
$$= 2\,\text{Arc}\sec\left(\frac{\sqrt[4]{x}}{2}\right) + C$$

4. a) $I = \displaystyle\int \frac{7t+26}{(t-2)(3t+4)}\,dt$

$$\frac{7t+26}{(t-2)(3t+4)} = \frac{A}{t-2} + \frac{B}{3t+4} = \frac{A(3t+4)+B(t-2)}{(t-2)(3t+4)}$$

$$7t + 26 = A(3t+4) + B(t-2)$$

si $t = 2$, $\quad 40 = 10A$, $\quad$ donc $A = 4$

si $t = \frac{-4}{3}$, $\quad \frac{50}{3} = \frac{-10}{3}B$, $\quad$ donc $B = -5$

$$I = \int \frac{4}{t-2}\,dt + \int \frac{-5}{3t+4}\,dt$$
$$= 4\ln\left|t-2\right| - \frac{5}{3}\ln\left|3t+4\right| + C$$

b) $I = \displaystyle\int \frac{2x^4+2x^3-2x^2-3x-2}{x^3+x^2-2x}\,dx$

En divisant, nous obtenons

$$\frac{2x^4+2x^3-2x^2-3x-2}{x^3+x^2-2x} = 2x + \frac{2x^2-3x-2}{x(x+2)(x-1)}$$

$$\frac{2x^2-3x-2}{x(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-1}$$

$$2x^2 - 3x - 2 = A(x+2)(x-1) + Bx(x-1) + Cx(x+2)$$

si $x = 0$, $\quad -2 = -2A$, $\quad$ donc $A = 1$
si $x = -2$, $\quad 12 = 6B$, $\quad$ donc $B = 2$
si $x = 1$, $\quad -3 = 3C$, $\quad$ donc $C = -1$

$$I = \int 2x\,dx + \int \frac{1}{x}\,dx + \int \frac{2}{x+2}\,dx + \int \frac{-1}{x-1}\,dx$$
$$= x^2 + \ln\left|x\right| + 2\ln\left|x+2\right| - \ln\left|x-1\right| + C$$

c) $I = \displaystyle\int \frac{6y^3+y^2-63}{y^4-81}\,dy$

$$\frac{6y^3+y^2-63}{y^4-81} = \frac{6y^3+y^2-63}{(y-3)(y+3)(y^2+9)}$$
$$= \frac{A}{y-3} + \frac{B}{y+3} + \frac{Cy+D}{y^2+9}$$

$$6y^3 + y^2 - 63 = A(y+3)(y^2+9) + B(y-3)(y^2+9) + (Cy+D)(y-3)(y+3)$$
$$= (A+B+C)y^3 + (3A-3B+D)y^2 + (9A+9B-9C)y + (27A-27B-9D)$$

$$A+B+C = 6 \quad\Rightarrow\quad A+B+C = 6 \quad ①$$
$$3A-3B+D = 1 \quad\Rightarrow\quad 3A-3B+D = 1 \quad ②$$
$$9A+9B-9C = 0 \quad\Rightarrow\quad A+B-C = 0 \quad ③$$
$$27A-27B-9D = -63 \quad\Rightarrow\quad 3A-3B-D = -7 \quad ④$$

4

② − ④, $2D = 8$, donc $D = 4$

① + ②, $2A + 2B = 6 \Rightarrow A + B = 3$ ⑤

En posant $D = 4$ dans ②

$3A - 3B + 4 = 1 \Rightarrow A - B = -1$ ⑥

⑤ + ⑥, $2A = 2$, donc $A = 1$

En posant $A = 1$ dans ⑤, $1 + B = 3$, donc $B = 2$

① − ③, $2C = 6$, donc $C = 3$

$$I = \int \frac{1}{y-3}\,dy + \int \frac{2}{y+3}\,dy + \int \frac{3y}{y^2+9}\,dy + \int \frac{4}{y^2+9}\,dy$$

$$= \ln|y-3| + 2\ln|y+3| + \frac{3}{2}\ln(y^2+9) + \frac{4}{3}\text{Arc tan}\left(\frac{y}{3}\right) + C$$

d) $I = \int \dfrac{10 + 2x^2 - 7x^3 + 9x}{x^3(2x+5)}\,dx$

$$\frac{10 + 2x^2 - 7x^3 + 9x}{x^3(2x+5)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x+5}$$

$$-7x^3 + 2x^2 + 9x + 10 = (2A + D)x^3 + (5A + 2B)x^2 + (5B + 2C)x + 5C$$

$2A + D = -7$ ①

$5A + 2B = 2$ ②

$5B + 2C = 9$ ③

$5C = 10$ ④, donc $C = 2$

En posant $C = 2$ dans ③, $5B + 4 = 9$, donc $B = 1$

En posant $B = 1$ dans ②, $5A + 2 = 2$, donc $A = 0$

En posant $A = 0$ dans ①, $D = -7$

$$I = \int \frac{1}{x^2}\,dx + \int \frac{2}{x^3}\,dx + \int \frac{-7}{2x+5}\,dx$$

$$= \frac{-1}{x} - \frac{1}{x^2} - \frac{7}{2}\ln|2x+5| + C$$

e) $I = \int \dfrac{3x^3 + 12x + 1}{(x^2+4)^2}\,dx$

$$\frac{3x^3 + 12x + 1}{(x^2+4)^2} = \frac{Ax + B}{x^2+4} + \frac{Cx + D}{(x^2+4)^2}$$

$$3x^3 + 12x + 1 = Ax^3 + Bx^2 + (4A + C)x + (4B + D)$$

$A = 3$

$B = 0$

$4A + C = 12$, ainsi $4(3) + C = 12$, donc $C = 0$

$4B + D = 1$, ainsi $4(0) + D = 1$, donc $D = 1$

$$I = \int \frac{3x}{x^2+4}\,dx + \int \frac{1}{(x^2+4)^2}\,dx = I_1 + I_2$$

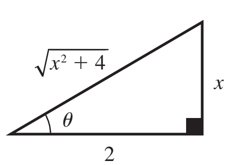
$$I_1 = \frac{3}{2}\ln(x^2+4) + C_1 \quad (u = x^2 + 4)$$

$$I_2 = \int \frac{1}{(x^2+4)^2}\,dx$$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\,d\theta$

$\theta = \text{Arc tan}\left(\dfrac{x}{2}\right)$



$$I_2 = \int \frac{2\sec^2\theta}{(4 + 4\tan^2\theta)^2}\,d\theta$$

$$= \frac{2}{16}\int \frac{\sec^2\theta}{\sec^4\theta}\,d\theta$$

$$= \frac{1}{8}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{8}\int \frac{1 + \cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{16}\left(\theta + \frac{\sin 2\theta}{2}\right) + C_2$$

$$= \frac{1}{16}(\theta + \sin\theta\cos\theta) + C_2$$

$$= \frac{1}{16}\left(\text{Arc tan}\left(\frac{x}{2}\right) + \frac{x}{\sqrt{x^2+4}} \cdot \frac{2}{\sqrt{x^2+4}}\right) + C_2$$

d'où $I = \dfrac{3}{2}\ln(x^2+4) + \dfrac{1}{16}\text{Arc tan}\left(\dfrac{x}{2}\right) + \dfrac{x}{8(x^2+4)} + C$

f) $I = \int \dfrac{\sin\theta}{\cos^2\theta - 7\cos\theta + 12}\,d\theta$

$u = \cos\theta$

$-du = \sin\theta\,d\theta$

$$= -\int \frac{1}{u^2 - 7u + 12}\,du$$

$$\frac{1}{u^2 - 7u + 12} = \frac{1}{(u-3)(u-4)} = \frac{A}{u-3} + \frac{B}{u-4}$$

$1 = A(u-4) + B(u-3)$

si $u = 4$, $1 = B$

si $u = 3$, $1 = -A$, donc $A = -1$

$$I = -\int \left[\frac{-1}{u-3} + \frac{1}{u-4}\right]du$$

$$= \ln|u-3| - \ln|u-4| + C$$

$$= \ln|\cos\theta - 3| - \ln|\cos\theta - 4| + C$$

g) $I = \int_0^1 \dfrac{3x^2 + 3x + 2}{(x+1)(x^2+1)}\,dx$

$$\frac{3x^2 + 3x + 2}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+1}$$

$$3x^2 + 3x + 2 = (A + B)x^2 + (B + C)x + (A + C)$$

$A + B = 3$ ①

$B + C = 3$ ②

$A + C = 2$ ③

① − ② $A - C = 0$ ④

④ + ③ $2A = 2$, donc $A = 1$

En posant $A = 1$ dans ①, $1 + B = 3$, donc $B = 2$

En posant $A = 1$ dans ③, $1 + C = 2$, donc $C = 1$

$$\frac{3x^2 + 3x + 2}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{1}{x^2+1}$$

$$I = \int_0^1 \frac{1}{x+1}\,dx + \int_0^1 \frac{2x}{x^2+1}\,dx + \int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \ln|x+1|\Big|_0^1 + \ln|x^2+1|\Big|_0^1 + \text{Arc tan}\,x\Big|_0^1$$

$$= (\ln 2 - \ln 1) + (\ln 2 - \ln 1) + (\text{Arc tan}\,1 - \text{Arc tan}\,0)$$

$$= 2\ln 2 + \frac{\pi}{4}$$

5. a) $I = \int \frac{x}{\sqrt{x+1}}\,dx$

i) Par parties

$u = x \qquad du = dx$

$dv = \frac{1}{\sqrt{x+1}}\,dx \qquad v = 2\sqrt{x+1}$

$I = 2x\sqrt{x+1} - 2\int \sqrt{x+1}\,dx$

$= 2x\sqrt{x+1} - \frac{4\sqrt{(x+1)^3}}{3} + C_1$

ii) Par changement de variable

$t = x + 1 \Rightarrow x = t - 1$
$dt = dx$

$I = \int \frac{t-1}{\sqrt{t}}\,dt$

$= \int \sqrt{t}\,dt - \int \frac{1}{\sqrt{t}}\,dt$

$= \frac{2}{3}t^{\frac{3}{2}} - 2\sqrt{t} + C_2$

$= \frac{2\sqrt{(x+1)^3}}{3} - 2\sqrt{x+1} + C_2$

iii) En i) $2x\sqrt{x+1} - \frac{4\sqrt{(x+1)^3}}{3} + C_1 = \sqrt{x+1}\left(\frac{2x-4}{3}\right) + C_1$

En ii) $\frac{2\sqrt{(x+1)^3}}{3} - 2\sqrt{x+1} + C_2 = \sqrt{x+1}\left(\frac{2x-4}{3}\right) + C_2$

Les deux réponses sont identiques, ainsi $C_1 = C_2$

b) $I = \int \sin 3x \cos 2x\,dx$

Première façon : par parties

$u = \sin 3x \qquad du = 3\cos 3x\,dx$

$dv = \cos 2x\,dx \qquad v = \frac{\sin 2x}{2}$

$I = \frac{\sin 3x \sin 2x}{2} - \frac{3}{2}\int \cos 3x \sin 2x\,dx$

$u = \cos 3x \qquad du = -3\sin 3x\,dx$

$dv = \sin 2x\,dx \qquad v = \frac{-\cos 2x}{2}$

$I = \frac{\sin 3x \sin 2x}{2} - \frac{3}{2}\left[\frac{-\cos 3x \cos 2x}{2} - \frac{3}{2}\int \sin 3x \cos 2x\,dx\right]$

$I = \frac{\sin 3x \sin 2x}{2} + \frac{3}{4}\cos 3x \cos 2x + \frac{9}{4}I$

$\frac{-5}{4}I = \frac{\sin 3x \sin 2x}{2} + \frac{3\cos 3x \cos 2x}{4} + C_1$

$I = \frac{-2\sin 3x \sin 2x - 3\cos 3x \cos 2x}{5} + C_2$

Deuxième façon : à l'aide de l'identité

$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

$I = \int \frac{1}{2}\left[\sin(3x - 2x) + \sin(3x + 2x)\right]dx$

$= \frac{1}{2}\int \sin x\,dx + \frac{1}{2}\int \sin 5x\,dx$

$= \frac{-\cos x}{2} - \frac{\cos 5x}{10} + C_3$

c) $I = \int \frac{x}{16 - x^2}\,dx$

Première façon : changement de variable

$u = 16 - x^2$
$du = -2x\,dx$

$I = \frac{-1}{2}\int \frac{1}{u}\,du$

$= \frac{-1}{2}\ln|u| + C$

$= \frac{-1}{2}\ln|16 - x^2| + C$

Deuxième façon : par décomposition en une somme de fractions partielles

$\frac{x}{16 - x^2} = \frac{A}{4 - x} + \frac{B}{4 + x}$

$x = A(4 + x) + B(4 - x)$

si $x = 4$, $4 = 8A$, donc $A = \frac{1}{2}$

si $x = -4$, $-4 = 8B$, donc $B = \frac{-1}{2}$

$I = \int \frac{\frac{1}{2}}{4 - x}\,dx + \int \frac{\frac{-1}{2}}{4 + x}\,dx$

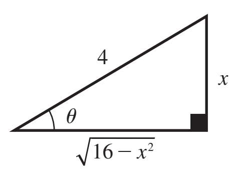
$= \frac{-1}{2}\ln|4 - x| - \frac{1}{2}\ln|4 + x| + C$

$= \frac{-1}{2}\left(\ln|4 - x| + \ln|4 + x|\right) + C$

$= \frac{-1}{2}\ln|16 - x^2| + C$

Troisième façon : substitution trigonométrique

$x = 4\sin\theta$
$dx = 4\cos\theta\,d\theta$



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$$\begin{aligned} I &= \int \frac{4\sin\theta}{16-16\sin^2\theta}(4\cos\theta)\,d\theta \\ &= \int \frac{\sin\theta\cos\theta}{(1-\sin^2\theta)}\,d\theta \\ &= \int \frac{\sin\theta}{\cos\theta}\,d\theta \\ &= \text{-}\ln|\cos\theta| + C_1 \\ &= \text{-}\ln\left|\frac{\sqrt{16-x^2}}{4}\right| + C_1 \\ &= \text{-}\ln(16-x^2)^{\frac{1}{2}} + \ln 4 + C_1 \\ &= \frac{\text{-}1}{2}\ln|16-x^2| + C \end{aligned}$$

d) $I = \int \sin^5 3\theta \cos^5 3\theta\, d\theta$

Première façon :

$$\begin{aligned} I &= \int \sin^5 3\theta\cos^4 3\theta\cos 3\theta\,d\theta \\ &= \int \sin^5 3\theta(\cos^2 3\theta)^2\cos 3\theta\,d\theta \\ &= \int \sin^5 3\theta(1-\sin^2 3\theta)^2\cos 3\theta\,d\theta \end{aligned}$$

$$\boxed{\begin{aligned} u &= \sin 3\theta \\ du &= 3\cos 3\theta\,d\theta \end{aligned}}$$

$$\begin{aligned} I &= \frac{1}{3}\int u^5(1-u^2)^2\,du \\ &= \frac{1}{3}\int u^5(1-2u^2+u^4)\,du \\ &= \frac{1}{3}\int(u^5-2u^7+u^9)\,du \\ &= \frac{1}{3}\left(\frac{u^6}{6}-\frac{2u^8}{8}+\frac{u^{10}}{10}\right)+C \\ &= \frac{\sin^6 3\theta}{18}-\frac{\sin^8 3\theta}{12}+\frac{\sin^{10} 3\theta}{30}+C_1 \end{aligned}$$

Deuxième façon (de façon analogue) :

$$I = \int \cos^5 3\theta(1-\cos^2 3\theta)^2\sin 3\theta\,d\theta$$

$$\boxed{\begin{aligned} u &= \cos 3\theta \\ du &= \text{-}3\sin 3\theta\,d\theta \end{aligned}}$$

$$\begin{aligned} I &= \frac{\text{-}1}{3}\int u^5(1-u^2)^2\,du \\ &= \frac{\text{-}1}{3}\left(\frac{u^6}{6}-\frac{2u^8}{8}+\frac{u^{10}}{10}\right)+C \\ &= \frac{\text{-}\cos^6 3\theta}{18}+\frac{\cos^8 3\theta}{12}-\frac{\cos^{10} 3\theta}{30}+C_2 \end{aligned}$$

Troisième façon :

$$\begin{aligned} I &= \int(\sin 3\theta\cos 3\theta)^5\,d\theta \\ &= \int\left(\frac{\sin 6\theta}{2}\right)^5 d\theta \\ &= \frac{1}{32}\int \sin^4 6\theta\sin 6\theta\,d\theta \\ &= \frac{1}{32}\int(1-\cos^2 6\theta)^2\sin 6\theta\,d\theta \end{aligned}$$

$$\boxed{\begin{aligned} u &= \cos 6\theta \\ du &= \text{-}6\sin 6\theta\,d\theta \end{aligned}}$$

$$\begin{aligned} I &= \frac{\text{-}1}{192}\int(1-u^2)^2\,du \\ &= \frac{\text{-}1}{192}\int(1-2u^2+u^4)\,du \\ &= \frac{\text{-}1}{192}\left(u-\frac{2u^3}{3}+\frac{u^5}{5}\right)+C \\ &= \frac{\text{-}\cos 6\theta}{192}+\frac{\cos^3 6\theta}{288}-\frac{\cos^5 6\theta}{960}+C_3 \end{aligned}$$

e) $I = 5\int \frac{1}{1-\cos x}\,dx$

Première façon : substitution trigonométrique

$$\boxed{u = \tan\left(\frac{x}{2}\right) \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad dx = \frac{2}{1+u^2}\,du}$$

$$\begin{aligned} I &= 5\int \frac{\frac{2}{1+u^2}}{1-\frac{1-u^2}{1+u^2}}\,du \\ &= 5\int \frac{1}{u^2}\,du \\ &= \frac{\text{-}5}{u}+C_1 \\ &= \frac{\text{-}5}{\tan\left(\frac{x}{2}\right)}+C_1 \end{aligned}$$

Deuxième façon : conjugué

$$\begin{aligned} I &= 5\int \frac{1}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right)dx \\ &= 5\int \frac{1+\cos x}{\sin^2 x}\,dx \\ &= 5\int\left(\frac{1}{\sin^2 x}+\frac{\cos x}{\sin^2 x}\right)dx \\ &= 5\left[\int \csc^2 x\,dx + \int \csc x\cot x\,dx\right] \\ &= 5(\text{-}\cot x - \csc x) + C_2 \\ &= \text{-}5\cot x - 5\csc x + C_2 \end{aligned}$$

6. a) $I = \int(5x^2+8)\ln x\,dx$

$$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}} \qquad \boxed{\begin{aligned} dv &= (5x^2+8)\,dx \\ v &= \frac{5x^3}{3}+8x \end{aligned}}$$

$$\begin{aligned} I &= \left(\frac{5x^3}{3}+8x\right)\ln x - \int\left(\frac{5x^3}{3}+8x\right)\frac{1}{x}\,dx \\ &= \left(\frac{5x^3}{3}+8x\right)\ln x - \frac{5}{3}\int x^2 dx - 8\int dx \\ &= \left(\frac{5x^3}{3}+8x\right)\ln x - \frac{5x^3}{9} - 8x + C \end{aligned}$$

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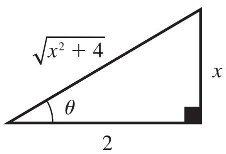
b) $I = \int \sin^2(5\theta)\cos^4(5\theta)\sin(5\theta)\,d\theta$
$= \int \left(1-\cos^2(5\theta)\right)\cos^4(5\theta)\sin(5\theta)\,d\theta$

$u = \cos(5\theta)$
$du = -5\sin(5\theta)\,d\theta$

$$I = \frac{-1}{5}\int (1-u^2)u^4\,du$$
$$= \frac{-1}{5}\int u^4\,du + \frac{1}{5}\int u^6\,du$$
$$= \frac{-u^5}{25} + \frac{u^7}{35} + C$$
$$= \frac{\cos^7(5\theta)}{35} - \frac{\cos^5(5\theta)}{25} + C$$

c) $I = \int x^2\sqrt{4+x^2}\,dx$

$x = 2\tan\theta$
$dx = 2\sec^2\theta\,d\theta$
$\theta = \text{Arc}\tan\left(\frac{x}{2}\right)$



$$I = \int 4\tan^2\theta\sqrt{4+4\tan^2\theta}\,2\sec^2\theta\,d\theta$$
$$= 16\int \tan^2\theta\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta$$
$$= 16\int \tan^2\theta\sec^3\theta\,d\theta$$
$$= 16\int (\sec^2\theta - 1)\sec^3\theta\,d\theta$$
$$= 16\int \sec^5\theta\,d\theta - 16\int \sec^3\theta\,d\theta$$
$$= 16\left[\frac{\sec^3\theta\tan\theta}{4} + \frac{3}{4}\int\sec^3\theta\,d\theta\right] - 16\int\sec^3\theta\,d\theta$$
$$= 4\sec^3\theta\tan\theta - 4\int\sec^3\theta\,d\theta$$
$$= 4\sec^3\theta\tan\theta - 4\left[\frac{\sec\theta\tan\theta + \ln\left|\sec\theta+\tan\theta\right|}{2}\right] + C$$
$$= 4\sec^3\theta\tan\theta - 2\sec\theta\tan\theta - 2\ln\left|\sec\theta+\tan\theta\right| + C$$
$$= 4\left(\frac{\sqrt{x^2+4}}{2}\right)^3\frac{x}{2} - 2\left(\frac{\sqrt{x^2+4}}{2}\right)\frac{x}{2} - 2\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right| + C$$
$$= \frac{x\sqrt{(x^2+4)^3}}{4} - \frac{x\sqrt{x^2+4}}{2} - 2\ln\left|\frac{\sqrt{x^2+4}+x}{2}\right| + C$$

d) $I = \int \frac{v\,\text{Arc sec}\,v}{\sqrt{v^2-1}}\,dv$

$u = \text{Arc sec}\,v$
$du = \frac{1}{v\sqrt{v^2-1}}\,dv$

$dz = \frac{v}{\sqrt{v^2-1}}\,dv$
$z = \sqrt{v^2-1}$

$$I = \sqrt{v^2-1}\,\text{Arc sec}\,v - \int\frac{\sqrt{v^2-1}}{v\sqrt{v^2-1}}\,dv$$
$$= \sqrt{v^2-1}\,\text{Arc sec}\,v - \int\frac{1}{v}\,dv$$
$$= \sqrt{v^2-1}\,\text{Arc sec}\,v - \ln|v| + C$$

e) $I = \int \frac{5x^3+4x^2+11x+4}{(x^2+1)^2}\,dx$

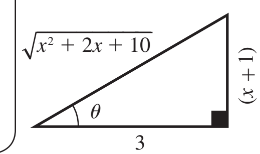
$$\frac{5x^3+4x^2+11x+4}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$5x^3+4x^2+11x+4 = Ax^3+Bx^2+(A+C)x+(B+D)$
$A = 5$
$B = 4$
$A + C = 11$, ainsi $5 + C = 11$, donc $C = 6$
$B + D = 4$, ainsi $4 + D = 4$, donc $D = 0$

$$I = \int\frac{5x}{x^2+1}\,dx + \int\frac{4}{x^2+1}\,dx + \int\frac{6x}{(x^2+1)^2}\,dx$$
$$= \frac{5}{2}\ln(x^2+1) + 4\,\text{Arc}\tan x - \frac{3}{x^2+1} + C$$

f) $I = \int\frac{x}{(x^2+2x+10)^{\frac{3}{2}}}\,dx = \int\frac{x}{\left[(x+1)^2+9\right]^{\frac{3}{2}}}\,dx$

$(x+1) = 3\tan\theta$
$x = 3\tan\theta - 1$
$dx = 3\sec^2\theta\,d\theta$
$\theta = \text{Arc}\tan\left(\frac{x+1}{3}\right)$



$$I = \int\frac{(3\tan\theta-1)3\sec^2\theta}{\left[9\tan^2\theta+9\right]^{\frac{3}{2}}}\,d\theta$$
$$= \frac{1}{9}\int\frac{(3\tan\theta-1)\sec^2\theta}{\left[\tan^2\theta+1\right]^{\frac{3}{2}}}\,d\theta$$
$$= \frac{1}{9}\int\frac{(3\tan\theta-1)\sec^2\theta}{\sec^3\theta}\,d\theta$$
$$= \frac{1}{9}\int\frac{3\tan\theta-1}{\sec\theta}\,d\theta$$
$$= \frac{1}{3}\int\frac{\tan\theta}{\sec\theta}\,d\theta - \frac{1}{9}\int\frac{1}{\sec\theta}\,d\theta$$
$$= \frac{1}{3}\int\sin\theta\,d\theta - \frac{1}{9}\int\cos\theta\,d\theta$$
$$= \frac{-\cos\theta}{3} - \frac{\sin\theta}{9} + C$$
$$= \frac{-1}{\sqrt{x^2+2x+10}} - \frac{(x+1)}{9\sqrt{x^2+2x+10}} + C$$

g) $I = \int e^t(\sin t + \cos t)\,dt$

$u = \sin t + \cos t$
$du = (\cos t - \sin t)\,dt$

$dv = e^t\,dt$
$v = e^t$

$$I = (\sin t + \cos t)e^t - \int e^t(\cos t - \sin t)\,dt$$

$u = \cos t - \sin t$
$du = (-\sin t - \cos t)\,dt$

$dv = e^t\,dt$
$v = e^t$

$$I = (\sin t + \cos t)e^t - (\cos t - \sin t)e^t - \int e^t(\sin t + \cos t)\,dt$$
$$I = (\sin t + \cos t)e^t - (\cos t - \sin t)e^t - I$$
$$2I = e^t\sin t + e^t\cos t - e^t\cos t + e^t\sin t + C_1$$
$$I = e^t\sin t + C$$

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h) $I = \int \frac{\sec^3 \sqrt{u} \tan^3 \sqrt{u}}{\sqrt{u}}\,du$

$= \int \sec^2 \sqrt{u} \tan^2 \sqrt{u} \frac{\sec \sqrt{u} \tan \sqrt{u}}{\sqrt{u}}\,du$

$= \int \sec^2 \sqrt{u} \left(\sec^2 \sqrt{u} - 1\right) \frac{\sec \sqrt{u} \tan \sqrt{u}}{\sqrt{u}}\,du$

$v = \sec \sqrt{u}$
$dv = \frac{\sec \sqrt{u} \tan \sqrt{u}}{2\sqrt{u}}\,du$

$I = 2\int u^2(u^2 - 1)\,du$

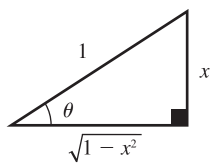
$= 2\int u^4\,du - 2\int u^2\,du$

$= \frac{2u^5}{5} - \frac{2u^3}{3} + C$

$= \frac{2\sec^5 \sqrt{u}}{5} - \frac{2\sec^3 \sqrt{u}}{3} + C$

i) $I = \int \frac{x^3 + x}{(1 - x^2)^2}\,dx$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$
$\theta = \text{Arc}\sin x$



$I = \int \frac{(\sin^3\theta + \sin\theta)}{(1 - \sin^2\theta)^2} \cos\theta\,d\theta$

$= \int \frac{(\sin^2\theta + 1)\cos\theta}{\cos^4\theta} \sin\theta\,d\theta$

$= \int \frac{(1 - \cos^2\theta + 1)}{\cos^3\theta} \sin\theta\,d\theta$

$= \int \frac{2 - \cos^2\theta}{\cos^3\theta} \sin\theta\,d\theta$

$= \int \frac{2 - u^2}{u^3}(-du)$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$= \int \frac{1}{u}\,du - \int \frac{2}{u^3}\,du$

$= \ln|u| + \frac{1}{u^2} + C$

$= \ln|\cos\theta| + \frac{1}{\cos^2\theta} + C$

$= \ln\sqrt{1 - x^2} + \frac{1}{1 - x^2} + C$

j) $I = \int \frac{x^4 + x^2 + 1}{x^3(x^2 + 4)}\,dx$

$\frac{x^4 + x^2 + 1}{x^3(x^2 + 4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx + E}{x^2 + 4}$

$x^4 + x^2 + 1 = (A + D)x^4 + (B + E)x^3 + (4A + C)x^2 + 4Bx + 4C$

① $A + D = 1$
② $B + E = 0$
③ $4A + C = 1$
④ $4B = 0$, donc $B = 0$
⑤ $4C = 1$, donc $C = \frac{1}{4}$

En posant $B = 0$ dans ②, $0 + E = 0$, donc $E = 0$

En posant $C = \frac{1}{4}$ dans ③, $4A + \frac{1}{4} = 1$, donc $A = \frac{3}{16}$

En posant $A = \frac{3}{16}$ dans ①, $\frac{3}{16} + D = 1$, donc $D = \frac{13}{16}$

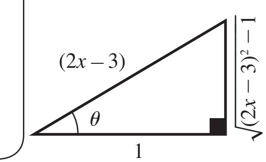
$I = \int \frac{\frac{3}{16}}{x}\,dx + \int \frac{\frac{1}{4}}{x^3}\,dx + \int \frac{\frac{13}{16}x}{x^2 + 4}\,dx$

$= \frac{3}{16}\ln|x| - \frac{1}{8x^2} + \frac{13}{32}\ln(x^2 + 4) + C$

k) $I = \int \frac{x + 1}{\sqrt{2x^2 - 6x + 4}}\,dx = \sqrt{2}\int \frac{x + 1}{\sqrt{4x^2 - 12x + 8}}\,dx$

$= \sqrt{2}\int \frac{x + 1}{\sqrt{(2x - 3)^2 - 1}}\,dx$

$(2x - 3) = \sec\theta$
$x = \frac{3 + \sec\theta}{2}$
$dx = \frac{\sec\theta\tan\theta}{2}\,d\theta$
$\theta = \text{Arc}\sec(2x - 3)$



$I = \sqrt{2}\int \frac{\left(\frac{3 + \sec\theta}{2} + 1\right)}{\sqrt{\sec^2\theta - 1}} \frac{\sec\theta\tan\theta}{2}\,d\theta$

$= \frac{\sqrt{2}}{4}\int \frac{(5 + \sec\theta)}{\tan\theta}\sec\theta\tan\theta\,d\theta$

$= \frac{\sqrt{2}}{4}\int 5\sec\theta\,d\theta + \frac{\sqrt{2}}{4}\int \sec^2\theta\,d\theta$

$= \frac{5\sqrt{2}}{4}\ln|\sec\theta + \tan\theta| + \frac{\sqrt{2}}{4}\tan\theta + C$

$= \frac{5\sqrt{2}}{4}\ln\left|(2x - 3) + \sqrt{(2x - 3)^2 - 1}\right| + \frac{\sqrt{2}}{4}\sqrt{(2x - 3)^2 - 1} + C$

l) $I = \int \frac{2 - \sin\theta}{2 + \sin\theta}\,d\theta$

$u = \tan\left(\frac{\theta}{2}\right)$ $\theta = 2\,\text{Arc}\tan u$
$\sin\theta = \frac{2u}{1 + u^2}$ $d\theta = \frac{2u}{1 + u^2}\,du$

$$I = \int \frac{2 - \frac{2u}{1+u^2}}{2 + \frac{2u}{1+u^2}} \frac{2}{1+u^2}\,du$$

$$= \int \frac{2 + 2u^2 - 2u}{2 + 2u^2 + 2u} \cdot \frac{2}{1+u^2}\,du$$

$$= 2\int \frac{u^2 - u + 1}{(u^2+u+1)(1+u^2)}\,du$$

$$\frac{u^2 - u + 1}{(u^2+u+1)(1+u^2)} = \frac{Au + B}{u^2+u+1} + \frac{Cu + D}{u^2+1}$$

$$u^2 - u + 1 = (A + C)u^3 + (B + C + D)u^2 + (A + C + D)u + (B + D)$$

① $A + C = 0$
② $B + C + D = 1$
③ $A + C + D = -1$
④ $B + D = 1$
② − ④ $C = 0$

En posant $C = 0$ dans ①, $A = 0$
En posant $A = 0$ et $C = 0$ dans ③, $D = -1$
En posant $D = -1$ dans ④, $B - 1 = 1$, donc $B = 2$

$$I = 2\int \left[\frac{2}{u^2+u+1} + \frac{-1}{u^2+1}\right] du$$

$$= 4\int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \frac{3}{4}}\,du - 2\int \frac{1}{u^2+1}\,du$$

$$= 16\int \frac{1}{(2u+1)^2 + 3}\,du - 2\text{ Arc tan } u + C$$

$$= \frac{16}{3}\frac{\sqrt{3}}{2}\text{ Arc tan}\left(\frac{2u+1}{\sqrt{3}}\right) - 2\text{ Arc tan } u + C$$

$$= \frac{8\sqrt{3}}{3}\text{ Arc tan}\left(\frac{2\tan\left(\frac{\theta}{2}\right) + 1}{\sqrt{3}}\right) - \theta + C$$

m) $I = \int e^{ax}\cos bx\,dx$

$u = e^{ax}$ $du = ae^{ax}\,dx$	$dv = \cos bx\,dx$ $v = \frac{\sin bx}{b}$

$$I = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\,dx$$

$u = e^{ax}$ $du = ae^{ax}\,dx$	$dv = \sin bx\,dx$ $v = \frac{-\cos bx}{b}$

$$I = \frac{e^{ax}\sin bx}{b} - \frac{a}{b}\left[\frac{-e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\,dx\right]$$

$$I = \frac{e^{ax}\sin bx}{b} + \frac{ae^{ax}\cos bx}{b^2} - \frac{a^2}{b^2}I$$

$$I\left(1 + \frac{a^2}{b^2}\right) = \frac{be^{ax}\sin bx + ae^{ax}\cos bx}{b^2} + C_1$$

$$I = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2 + b^2} + C$$

n) $I = \int ax\sin bx\,dx$

$u = ax$ $du = a\,dx$	$dv = \sin bx\,dx$ $v = \frac{-\cos bx}{b}$

$$I = \frac{-ax\cos bx}{b} + \frac{a}{b}\int \cos bx\,dx$$

$$= \frac{-ax\cos bx}{b} + \frac{a\sin bx}{b^2} + C$$

o) $I = \int \frac{(2e^x + 1)}{(e^x - 2)^2}\,dx$

$u = e^x - 2 \Rightarrow e^x = u + 2$ $du = e^x\,dx \;\Rightarrow dx = \frac{1}{e^x}du = \frac{1}{u+2}du$

$$I = \int \frac{(2(u+2)+1)}{u^2}\frac{1}{(u+2)}\,du = \int \frac{2u+5}{u^2(u+2)}\,du$$

$$\frac{2u+5}{u^2(u+2)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+2}$$

$$2u + 5 = (A + C)u^2 + (2A + B)u + 2B$$

① $A + C = 0$
② $2A + B = 2$
③ $2B = 5$, donc $B = \frac{5}{2}$

En posant $B = \frac{5}{2}$ dans ②, $A = \frac{-1}{4}$

En posant $A = \frac{-1}{4}$ dans ①, $C = \frac{1}{4}$

$$I = \int \frac{\frac{-1}{4}}{u}\,du + \int \frac{\frac{5}{2}}{u^2}\,du + \int \frac{\frac{1}{4}}{u+2}\,du$$

$$= \frac{-1}{4}\ln|u| - \frac{5}{2u} + \frac{1}{4}\ln|u+2| + C$$

$$= \frac{-\ln|e^x - 2|}{4} - \frac{5}{2(e^x - 2)} + \frac{\ln(e^x)}{4} + C$$

$$= \frac{x}{4} - \frac{\ln|e^x - 2|}{4} - \frac{5}{2(e^x - 2)} + C$$

7. a) $\int_2^3 \frac{x}{x^2 - 1}\,dx = \frac{1}{2}\int_3^8 \frac{1}{u}\,du \qquad (u = x^2 - 1)$

$$= \frac{1}{2}\ln|u|\Big|_3^8$$

$$= \frac{1}{2}(\ln 8 - \ln 3)$$

$$= \frac{1}{2}\ln\left(\frac{8}{3}\right)$$

4

b) $\displaystyle\int_0^1 \frac{1}{x^3+3x^2+3x+1}\,dx = \int_0^1 \frac{1}{(x+1)^3}\,dx$

$\displaystyle = \int_1^2 \frac{1}{u^3}\,du \qquad (u = x+1)$

$\displaystyle = \left.\frac{-1}{2u^2}\right|_1^2$

$\displaystyle = \frac{3}{8}$

c) $I = \int (\text{Arc}\sin x)^2\,dx$

$u = (\text{Arc}\sin x)^2$, $du = \dfrac{2\,\text{Arc}\sin x}{\sqrt{1-x^2}}\,dx$

$dv = dx$, $v = x$

$\displaystyle I = x(\text{Arc}\sin x)^2 - 2\int \frac{x\,\text{Arc}\sin x}{\sqrt{1-x^2}}\,dx$

$u = \text{Arc}\sin x$, $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$

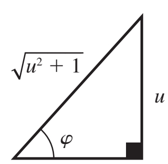
$dv = \dfrac{x}{\sqrt{1-x^2}}\,dx$, $v = \text{-}\sqrt{1-x^2}$

$\displaystyle I = x(\text{Arc}\sin x)^2 - 2\left[\text{-}(\text{Arc}\sin x)\sqrt{1-x^2} + \int 1\,dx\right]$

$= x(\text{Arc}\sin x)^2 + 2\sqrt{1-x^2}\,\text{Arc}\sin x - 2x + C$

d) $\displaystyle I = \int \frac{\cos\theta}{\sin\theta\sqrt{1+\sin^2\theta}}\,d\theta = \int \frac{1}{u\sqrt{1+u^2}}\,du \qquad (u = \sin\theta)$

$u = \tan\varphi$
$du = \sec^2\varphi\,d\varphi$
$\varphi = \text{Arc}\tan u$



$\displaystyle I = \int \frac{\sec^2\varphi}{\tan\varphi\sqrt{1+\tan^2\varphi}}\,d\varphi$

$\displaystyle = \int \frac{\sec^2\varphi}{\tan\varphi\sec\varphi}\,d\varphi$

$= \int \csc\varphi\,d\varphi$

$= \ln|\csc\varphi - \cot\varphi| + C$

$\displaystyle = \ln\left|\frac{\sqrt{1+u^2}}{u} - \frac{1}{u}\right| + C$

$\displaystyle = \ln\left|\frac{\sqrt{1+\sin^2\theta}-1}{\sin\theta}\right| + C$

e) $\displaystyle I = \int \frac{\ln t}{\sqrt{t}}\,dt$

$u = \ln t$, $du = \dfrac{1}{t}\,dx$

$dv = \dfrac{1}{\sqrt{t}}\,dx$, $v = 2\sqrt{t}$

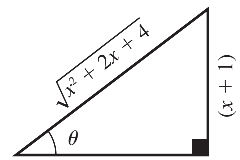
$\displaystyle I = 2\sqrt{t}\ln t - 2\int t^{\frac{-1}{2}}\,dt$

$= 2\sqrt{t}\ln t - 4\sqrt{t} + C$

d'où $\displaystyle\int_1^4 \frac{\ln t}{\sqrt{t}}\,dt = \left(2\sqrt{t}\ln t - 4\sqrt{t}\right)\Big|_1^4 = 4\ln 4 - 4$

f) $\displaystyle I = \int \frac{x^2+2x+1}{(x^2+2x+4)^{\frac{3}{2}}}\,dx = \int \frac{(x+1)^2}{\left((x+1)^2+3\right)^{\frac{3}{2}}}\,dx$

$x + 1 = \sqrt{3}\tan\theta$
$dx = \sqrt{3}\sec^2\theta\,d\theta$
$\theta = \text{Arc}\tan\left(\dfrac{x+1}{\sqrt{3}}\right)$



$\displaystyle I = \int \frac{3\tan^2\theta\,\sqrt{3}\sec^2\theta}{(3\tan^2\theta+3)^{\frac{3}{2}}}\,d\theta$

$\displaystyle = \int \frac{\tan^2\theta\sec^2\theta}{\sec^3\theta}\,d\theta$

$\displaystyle = \int \frac{\tan^2\theta}{\sec\theta}\,d\theta$

$\displaystyle = \int \frac{\sec^2\theta - 1}{\sec\theta}\,d\theta$

$= \int \sec\theta\,d\theta - \int \cos\theta\,d\theta$

$= \ln|\sec\theta + \tan\theta| - \sin\theta + C_1$

$\displaystyle = \ln\left|\frac{\sqrt{x^2+2x+4}}{\sqrt{3}} + \frac{x+1}{\sqrt{3}}\right| - \frac{x+1}{\sqrt{x^2+2x+4}} + C_1$

$\displaystyle = \ln\left|\sqrt{x^2+2x+4} + x + 1\right| - \frac{x+1}{\sqrt{x^2+2x+4}} + C$

g) $\displaystyle I = \int \frac{\sin x}{(1+\sin x)^2}\,dx$

$u = \tan\left(\dfrac{x}{2}\right) \qquad \sin x = \dfrac{2u}{1+u^2} \qquad dx = \dfrac{2}{1+u^2}\,du$

$\displaystyle I = \int \frac{\frac{2u}{1+u^2}}{\left(1+\frac{2u}{1+u^2}\right)^2}\,\frac{2}{1+u^2}\,du$

$\displaystyle = 4\int \frac{u}{(u^2+2u+1)^2}\,du$

$\displaystyle = 4\int \frac{u}{(u+1)^2}\,du$

$\displaystyle = 4\int \frac{v-1}{v^2}\,dv \qquad (v = u+1 \Rightarrow u = v-1)$

$\displaystyle = 4\left[\int \frac{1}{v}\,dv - \int \frac{1}{v^2}\,dv\right]$

$\displaystyle = 4\ln|v| + \frac{4}{v} + C$

$\displaystyle = 4\ln|u+1| + \frac{4}{(u+1)} + C$

$\displaystyle = 4\ln\left|1+\tan\left(\frac{x}{2}\right)\right| + \frac{4}{1+\tan\left(\frac{x}{2}\right)} + C$

h) $\displaystyle\int_1^4 \frac{1}{2+\sqrt{y}}\,dy = \int_3^4 \frac{2u-4}{u}\,du$

$\displaystyle = 2\int_3^4 du - 4\int_3^4 \frac{1}{u}\,du$

$\displaystyle = 2u\Big|_3^4 - 4\ln|u|\Big|_3^4$

$\displaystyle = 2 + 4\ln\left(\frac{3}{4}\right)$

$u = 2+\sqrt{y}$
$du = \dfrac{1}{2\sqrt{y}}\,dy$
$dy = 2\sqrt{y}\,du$
$dy = 2(u-2)\,du$

i) $\displaystyle \frac{2x^3-8x^2+9x+1}{(x-2)^2} = 2x + \frac{(x+1)}{(x-2)^2}$

$\displaystyle = 2x + \frac{A}{(x-2)} + \frac{B}{(x-2)^2}$

$\displaystyle = 2x + \frac{1}{(x-2)} + \frac{3}{(x-2)^2}$

$\displaystyle \int_0^1 \frac{2x^3-8x^2+9x+1}{(x-2)^2}\,dx$

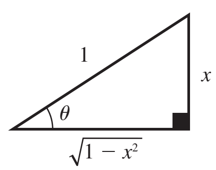
$\displaystyle = 2\int_0^1 x\,dx + \int_0^1 \frac{1}{(x-2)}\,dx + 3\int_0^1 \frac{1}{(x-2)^2}\,dx$

$\displaystyle = x^2\Big|_0^1 + \ln|x-2|\Big|_0^1 - \frac{3}{(x-2)}\Bigg|_0^1$

$\displaystyle = \frac{5}{2} - \ln 2$

j) $\displaystyle I = \int \frac{16x^4}{\sqrt{1-x^2}}\,dx$

$x = \sin\theta$
$dx = \cos\theta\,d\theta$
$\theta = \text{Arc}\sin x$



$\displaystyle I = 16\int \frac{\sin^4\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta$

$\displaystyle = 16\int \sin^4\theta\,d\theta$

$\displaystyle = 16\left[\frac{-\sin^3\theta\cos\theta}{4} + \frac{3}{4}\int \sin^2\theta\,d\theta\right]$

$\displaystyle = -4\sin^3\theta\cos\theta + 12\left[\frac{-\sin\theta\cos\theta}{2} + \frac{1}{2}\int d\theta\right]$

$= -4\sin^3\theta\cos\theta - 6\sin\theta\cos\theta + 6\theta + C$

$= -4x^3\sqrt{1-x^2} - 6x\sqrt{1-x^2} + 6\,\text{Arc}\sin x + C$

k) $I = \int \cos\sqrt{x}\,dx = 2\int z\cos z\,dz \quad \left(z = \sqrt{x} \text{ et } dx = 2z\,dz\right)$

$u = z$
$du = dz$

$dv = \cos z\,dz$
$v = \sin z$

$I = 2\left[z\sin z - \int \sin z\,dz\right]$

$= 2(z\sin z + \cos z) + C$

$= 2\left[\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}\right] + C$

l) $\displaystyle \int_0^3 \frac{t}{\sqrt{t+1}}\,dt = 2\int_1^2 (u^2-1)\,du$

$\displaystyle = 2\left(\frac{u^3}{3} - u\right)\Bigg|_1^2$

$\displaystyle = \frac{8}{3}$

$u = \sqrt{t+1}$
$t = u^2 - 1$
$dt = 2u\,du$

m) $\displaystyle I = \int \frac{x}{\sqrt{ax+b}}\,dx$

$u = x$
$du = dx$

$dv = \dfrac{1}{\sqrt{ax+b}}\,dx$
$v = \dfrac{2\sqrt{ax+b}}{a}$

$\displaystyle I = \frac{2x\sqrt{ax+b}}{a} - \frac{2}{a}\int (ax+b)^{\frac{1}{2}}\,dx$

$\displaystyle = \frac{2x\sqrt{ax+b}}{a} - \frac{4\sqrt{(ax+b)^3}}{3a^2} + C$

n) $I = \int \sin ax\cos bx\,dx$

$u = \sin ax$
$du = a\cos ax\,dx$

$dv = \cos bx\,dx$
$v = \dfrac{\sin bx}{b}$

$\displaystyle I = \frac{\sin ax\sin bx}{b} - \frac{a}{b}\int \cos ax\sin bx\,dx$

$u = \cos ax$
$du = -a\sin ax\,dx$

$dv = \sin bx\,dx$
$v = \dfrac{-\cos bx}{b}$

$\displaystyle I = \frac{\sin ax\sin bx}{b} - \frac{a}{b}\left[\frac{-\cos ax\cos bx}{b} - \frac{a}{b}\int \sin ax\cos bx\,dx\right]$

$\displaystyle = \frac{\sin ax\sin bx}{b} + \frac{a\cos ax\cos bx}{b^2} + \frac{a^2}{b^2}I$

$\displaystyle I\left(1 - \frac{a^2}{b^2}\right) = \frac{\sin ax\sin bx}{b} + \frac{a\cos ax\cos bx}{b^2} + C_1$

$\displaystyle I = \frac{b\sin ax\sin bx + a\cos ax\cos bx}{b^2-a^2} + C$

o) $\displaystyle I = \int \frac{7}{4\sin x - 3\cos x}\,dx$

$u = \tan\left(\dfrac{x}{2}\right)$ $\cos x = \dfrac{1-u^2}{1+u^2}$
$\sin x = \dfrac{2u}{1+u^2}$ $dx = \dfrac{2}{1+u^2}\,du$

$\displaystyle I = 7\int \frac{1}{4\left(\frac{2u}{1+u^2}\right) - 3\left(\frac{1-u^2}{1+u^2}\right)}\,\frac{2}{1+u^2}\,du$

$\displaystyle I = 14\int \frac{1}{8u-3+3u^2}\,du$

$\displaystyle \frac{1}{3u^2+8u-3} = \frac{1}{(3u-1)(u+3)} = \frac{A}{3u-1} + \frac{B}{u+3}$

$1 = A(u+3) + B(3u-1)$

En posant $u = -3$, $B = \dfrac{-1}{10}$

En posant $u = \dfrac{1}{3}$, $A = \dfrac{3}{10}$

$$I = 14\left[\int \frac{\frac{3}{10}}{3u-1}\,du + \int \frac{\frac{-1}{10}}{u+3}\,du\right]$$
$$= 14\left[\frac{\ln|3u-1|}{10} - \frac{\ln|u+3|}{10}\right] + C$$
$$= \frac{7}{5}\ln\left|\frac{3u-1}{u+3}\right| + C$$
$$= \frac{7}{5}\ln\left|\frac{3\tan\left(\frac{x}{2}\right)-1}{\tan\left(\frac{x}{2}\right)+3}\right| + C$$

8. a) i) $\int x^n \cos x\,dx$

$u = x^n$, $du = nx^{n-1}\,dx$ | $dv = \cos x\,dx$, $v = \sin x$

$$\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$$

ii) $\int x^n \sin x\,dx$

$u = x^n$, $du = nx^{n-1}\,dx$ | $dv = \sin x\,dx$, $v = \text{-}\cos x$

$$\int x^n \sin x\,dx = \text{-}x^n\cos x + n\int x^{n-1}\cos x\,dx$$

iii) $$\int \tan^n(ax)\,dx = \int \tan^{n-2}(ax)\tan^2(ax)\,dx$$
$$= \int \tan^{n-2}(ax)\left(\sec^2(ax)-1\right)dx$$
$$= \int \tan^{n-2}(ax)\sec^2(ax)dx - \int \tan^{n-2}(ax)\,dx$$
$$= \frac{1}{a}\int u^{n-2}\,du - \int \tan^{n-2}(ax)\,dx \quad (u = \tan(ax))$$
$$= \frac{1}{a}\,\frac{u^{n-1}}{n-1} - \int \tan^{n-2}(ax)\,dx$$
$$= \frac{\tan^{n-1}(ax)}{a(n-1)} - \int \tan^{n-2}(ax)\,dx$$

iv) $\int x^k(\ln x)^n\,dx$, où $k \neq \text{-}1$

$u = (\ln x)^n$, $du = \frac{n(\ln x)^{n-1}}{x}\,dx$ | $dv = x^k dx$, $v = \frac{x^{k+1}}{k+1}$, car $k \neq \text{-}1$

$$\int x^k(\ln x)^n\,dx = \frac{x^{k+1}(\ln x)^n}{k+1} - \frac{n}{k+1}\int x^k(\ln x)^{n-1}\,dx,$$
où $k \neq \text{-}1$

b) i) $I = \int x^3 \sin x\,dx$
$$I = \text{-}x^3\cos x + 3\int x^2\cos x\,dx \quad \text{(de ii))}$$
$$= \text{-}x^3\cos x + 3\left[x^2\sin x - 2\int x\sin x\,dx\right] \quad \text{(de i))}$$
$$= \text{-}x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$
$$= \text{-}x^3\cos x + 3x^2\sin x - 6\left[\text{-}x\cos x + \int \cos x\,dx\right] \quad \text{(de ii))}$$
$$= \text{-}x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

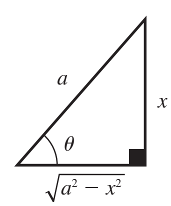
ii) $I = \int x^9(\ln x)^4\,dx$ (formule iv))
$$I = \frac{x^{10}(\ln x)^4}{10} - \frac{4}{10}\int x^9(\ln x)^3\,dx$$
$$= \frac{x^{10}(\ln x)^4}{10} - \frac{2}{5}\left[\frac{x^{10}(\ln x)^3}{10} - \frac{3}{10}\int x^9(\ln x)^2\,dx\right]$$
$$= \frac{x^{10}(\ln x)^4}{10} - \frac{x^{10}(\ln x)^3}{25} + \frac{3}{25}\int x^9(\ln x)^2\,dx$$
$$= \frac{x^{10}(\ln x)^4}{10} - \frac{x^{10}(\ln x)^3}{25} + \frac{3}{25}\left[\frac{x^{10}(\ln x)^2}{10} - \frac{2}{10}\int x^9(\ln x)\,dx\right]$$
$$= \frac{x^{10}(\ln x)^4}{10} - \frac{x^{10}(\ln x)^3}{25} + \frac{3x^{10}(\ln x)^2}{250} - \frac{3}{125}\int x^9(\ln x)\,dx$$
$$= \frac{x^{10}(\ln x)^4}{10} - \frac{x^{10}(\ln x)^3}{25} + \frac{3x^{10}(\ln x)^2}{250} - \frac{3}{125}\left[\frac{x^{10}(\ln x)}{10} - \frac{1}{10}\int x^9\,dx\right]$$
$$= \frac{x^{10}(\ln x)^4}{10} - \frac{x^{10}(\ln x)^3}{25} + \frac{3x^{10}(\ln x)^2}{250} - \frac{3x^{10}(\ln x)}{1250} + \frac{3x^{10}}{12\,500} + C$$

iii) $I = \int \tan^4(5x)\,dx$ (formule iii))
$$I = \frac{\tan^3(5x)}{5(3)} - \int \tan^2(5x)\,dx$$
$$= \frac{\tan^3(5x)}{15} - \left[\frac{\tan(5x)}{5} - \int 1\,dx\right]$$
$$= \frac{\tan^3(5x)}{15} - \frac{\tan(5x)}{5} + x + C$$

iv) $I = \int \tan^5(4x)\,dx$
$$I = \frac{\tan^4(4x)}{4(4)} - \int \tan^3(4x)\,dx$$
$$= \frac{\tan^4(4x)}{16} - \left[\frac{\tan^2(4x)}{4(2)} - \int \tan(4x)\,dx\right]$$
$$= \frac{\tan^4(4x)}{16} - \frac{\tan^2(4x)}{8} - \frac{\ln|\cos 4x|}{4} + C$$

9. a) i) $I = \int x^2\sqrt{a^2-x^2}\,dx$

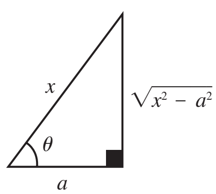
$x = a\sin\theta$
$dx = a\cos\theta\,d\theta$
$\theta = \text{Arc}\sin\left(\frac{x}{a}\right)$



$$\begin{aligned}
I &= \int a^2 \sin^2\theta \sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta\, d\theta \\
&= a^4 \int \sin^2\theta \cos^2\theta\, d\theta \\
&= a^4 \int \sin^2\theta\,(1 - \sin^2\theta)\, d\theta \\
&= a^4 \left[\int \sin^2\theta\, d\theta - \int \sin^4\theta\, d\theta\right] \\
&= a^4 \left\{\left[\frac{-\sin\theta\cos\theta}{2} + \frac{1}{2}\int d\theta\right] - \left[\frac{-\sin^3\theta\cos\theta}{4} + \frac{3}{4}\int \sin^2\theta\, d\theta\right]\right\} + C \\
&= a^4 \left\{\left[\frac{-\sin\theta\cos\theta}{2} + \frac{\theta}{2}\right] - \left[\frac{-\sin^3\theta\cos\theta}{4} + \frac{3}{4}\left(\frac{-\sin\theta\cos\theta}{2} + \frac{\theta}{2}\right)\right]\right\} + C \\
&= a^4 \left\{\frac{\sin^3\theta\cos\theta}{4} - \frac{\sin\theta\cos\theta}{8} + \frac{\theta}{8}\right\} + C \\
&= a^4 \left\{\frac{1}{4}\left(\frac{x}{a}\right)^3\left(\frac{\sqrt{a^2 - x^2}}{a}\right) - \frac{1}{8}\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2 - x^2}}{a}\right) + \frac{1}{8}\,\text{Arc sin}\left(\frac{x}{a}\right)\right\} + C \\
&= \frac{x^3}{4}\sqrt{a^2 - x^2} - \frac{a^2 x}{8}\sqrt{a^2 - x^2} + \frac{a^4}{8}\,\text{Arc sin}\left(\frac{x}{a}\right) + C \\
&= \frac{x}{8}(2x^2 - a^2)\sqrt{a^2 - x^2} + \frac{a^4}{8}\,\text{Arc sin}\left(\frac{x}{a}\right) + C
\end{aligned}$$

ii) $I = \int x^2 \sqrt{x^2 - a^2}\, dx$

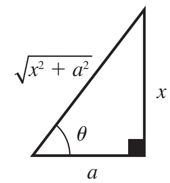
$$\begin{aligned}
x &= a\sec\theta \\
dx &= a\sec\theta\tan\theta\, d\theta \\
\theta &= \text{Arc sec}\left(\frac{x}{a}\right)
\end{aligned}$$



$$\begin{aligned}
I &= \int a^2 \sec^2\theta \sqrt{a^2\sec^2\theta - a^2}\; a\sec\theta\tan\theta\, d\theta \\
&= a^4 \int \sec^3\theta \tan^2\theta\, d\theta \\
&= a^4 \int \sec^3\theta\,(\sec^2\theta - 1)\, d\theta \\
&= a^4 \left[\int \sec^5\theta\, d\theta - \int \sec^3\theta\, d\theta\right] \\
&= a^4 \left[\left(\frac{\sec^3\theta\tan\theta}{4} + \frac{3}{4}\int \sec^3\theta\, d\theta\right) - \int \sec^3\theta\, d\theta\right] \\
&= a^4 \left[\frac{\sec^3\theta\tan\theta}{4} - \frac{1}{4}\int \sec^3\theta\, d\theta\right] \\
&= a^4 \left[\frac{\sec^3\theta\tan\theta}{4} - \frac{1}{4}\left(\frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2}\right)\right] + C_1 \\
&= \frac{a^4\sec^3\theta\tan\theta}{4} - \frac{a^4\sec\theta\tan\theta}{8} - \frac{a^4\ln|\sec\theta + \tan\theta|}{8} + C_1 \\
&= \frac{a^4}{4}\left(\frac{x}{a}\right)^3\frac{\sqrt{x^2 - a^2}}{a} - \frac{a^4}{8}\left(\frac{x}{a}\right)\frac{\sqrt{x^2 - a^2}}{a} - \frac{a^4}{8}\ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C_1 \\
&= \frac{x^3}{4}\sqrt{x^2 - a^2} - \frac{a^2 x}{8}\sqrt{x^2 - a^2} - \frac{a^4}{8}\ln\left|x + \sqrt{x^2 - a^2}\right| + C \\
&= \frac{x}{8}(2x^2 - a^2)\sqrt{x^2 - a^2} - \frac{a^4}{8}\ln\left|x + \sqrt{x^2 - a^2}\right| + C
\end{aligned}$$

iii) $I = \int \frac{\sqrt{x^2 + a^2}}{x^2}\, dx$

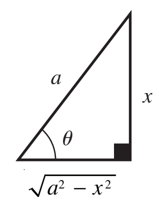
$$\begin{aligned}
x &= a\tan\theta \\
dx &= a\sec^2\theta\, d\theta \\
\theta &= \text{Arc tan}\left(\frac{x}{a}\right)
\end{aligned}$$



$$\begin{aligned}
I &= \int \frac{\sqrt{a^2\tan^2\theta + a^2}\; a\sec^2\theta}{a^2\tan^2\theta}\, d\theta \\
&= \int \frac{\sec^3\theta}{\tan^2\theta}\, d\theta \\
&= \int \frac{\sec\theta\sec^2\theta}{\tan^2\theta}\, d\theta \\
&= \int \frac{\sec\theta\,(1 + \tan^2\theta)}{\tan^2\theta}\, d\theta \\
&= \int \frac{\sec\theta}{\tan^2\theta}\, d\theta + \int \frac{\sec\theta\tan^2\theta}{\tan^2\theta}\, d\theta \\
&= \int \csc\theta\cot\theta\, d\theta + \int \sec\theta\, d\theta \\
&= -\csc\theta + \ln|\sec\theta + \tan\theta| + C_1 \\
&= \frac{-\sqrt{x^2 + a^2}}{x} + \ln\left|\frac{\sqrt{x^2 + a^2}}{a} + \frac{x}{a}\right| + C_1 \\
&= \frac{-\sqrt{x^2 + a^2}}{x} + \ln\left|\sqrt{x^2 + a^2} + x\right| + C
\end{aligned}$$

iv) $I = \int \frac{\sqrt{a^2 - x^2}}{x^2}\, dx$

$$\begin{aligned}
x &= a\sin\theta \\
dx &= a\cos\theta\, d\theta \\
\theta &= \text{Arc sin}\left(\frac{x}{a}\right)
\end{aligned}$$

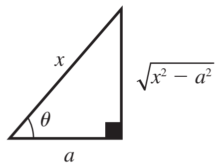


$$\begin{aligned}
I &= \int \frac{\sqrt{a^2 - a^2\sin^2\theta}\; a\cos\theta}{a^2\sin^2\theta}\, d\theta \\
&= \int \frac{\cos^2\theta}{\sin^2\theta}\, d\theta \\
&= \int \cot^2\theta\, d\theta \\
&= \int (\csc^2\theta - 1)\, d\theta \\
&= -\cot\theta - \theta + C \\
&= \frac{-\sqrt{a^2 - x^2}}{x} - \text{Arc sin}\left(\frac{x}{a}\right) + C
\end{aligned}$$

4

v) $I = \int (x^2 - a^2)^{\frac{3}{2}}\,dx$

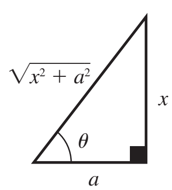
$$x = a\sec\theta \qquad dx = a\sec\theta\tan\theta\,d\theta \qquad \theta = \text{Arc sec}\left(\frac{x}{a}\right)$$



$$\begin{aligned}
I &= \int (a^2\sec^2\theta - a^2)^{\frac{3}{2}}\, a\sec\theta\tan\theta\,d\theta\\
&= a^4\int (\tan^2\theta)^{\frac{3}{2}}\sec\theta\tan\theta\,d\theta\\
&= a^4\int \sec\theta\tan^4\theta\,d\theta\\
&= a^4\int \sec\theta(\sec^2\theta - 1)^2\,d\theta\\
&= a^4\int \sec\theta(\sec^4\theta - 2\sec\theta + 1)\,d\theta\\
&= a^4\left\{\int \sec^5\theta\,d\theta - 2\int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta\right\}\\
&= a^4\left\{\frac{\sec^3\theta\tan\theta}{4} + \frac{3}{4}\int\sec^3\theta\,d\theta - 2\int\sec^3\theta\,d\theta + \ln\left|\sec\theta + \tan\theta\right|\right\}\\
&= a^4\left\{\frac{\sec^3\theta\tan\theta}{4} - \frac{5}{4}\int\sec^3\theta\,d\theta + \ln\left|\sec\theta\tan\theta\right|\right\}\\
&= a^4\left\{\frac{\sec^3\theta\tan\theta}{4} - \frac{5}{4}\left[\frac{\sec\theta\tan\theta + \ln\left|\sec\theta\tan\theta\right|}{2}\right] + \ln\left|\sec\theta + \tan\theta\right|\right\} + C_1\\
&= a^4\left\{\frac{\sec^3\theta\tan\theta}{4} - \frac{5\sec\theta\tan\theta}{8} + \frac{3}{8}\ln\left|\sec\theta + \tan\theta\right|\right\} + C_1\\
&= a^4\left\{\frac{1}{4}\left(\frac{x}{a}\right)^3\left(\frac{\sqrt{x^2-a^2}}{a}\right) - \frac{5}{8}\left(\frac{x}{a}\right)\left(\frac{\sqrt{x^2-a^2}}{a}\right) + \frac{3}{8}\ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right|\right\} + C_1\\
&= \frac{x}{8}(2x^2 - 5a^2)\sqrt{x^2-a^2} + \frac{3a^4}{8}\ln\left|x + \sqrt{x^2-a^2}\right| + C
\end{aligned}$$

vi) $I = \int \frac{\sqrt{x^2 + a^2}}{x}\,dx$

$$x = a\tan\theta \qquad dx = a\sec^2\theta\,d\theta \qquad \theta = \text{Arc tan}\left(\frac{x}{a}\right)$$



$$\begin{aligned}
I &= \int \frac{\sqrt{a^2\tan^2\theta + a^2}}{a\tan\theta}\,a\sec^2\theta\,d\theta\\
&= a\int\frac{\sec^3\theta}{\tan\theta}\,d\theta\\
&= a\int\frac{\sec\theta\sec^2\theta}{\tan\theta}\,d\theta\\
&= a\int\frac{\sec\theta\,(\tan^2\theta + 1)}{\tan\theta}\,d\theta\\
&= a\left[\int\sec\theta\tan\theta\,d\theta + \int\frac{\sec\theta}{\tan\theta}\,d\theta\right]\\
&= a\left[\int\sec\theta\tan\theta\,d\theta + \int\csc\theta\,d\theta\right]\\
&= a\left[\sec\theta - \ln\left|\csc\theta + \cot\theta\right|\right] + C\\
&= a\left[\frac{\sqrt{x^2+a^2}}{a} - \ln\left|\frac{\sqrt{x^2+a^2}}{x} + \frac{a}{x}\right|\right] + C\\
&= \sqrt{x^2+a^2} - a\ln\left|\frac{a + \sqrt{x^2+a^2}}{x}\right| + C
\end{aligned}$$

b) i) $I = \frac{1}{7}\int\frac{\sqrt{5 - x^2}}{x^2}\,dx$

Formule iv), où $a^2 = 5$

$$I = \frac{-\sqrt{5-x^2}}{7x} - \frac{1}{7}\text{Arc sin}\left(\frac{x}{\sqrt{5}}\right) + C$$

ii) $I = \int\frac{\sqrt{x^2+4}}{x}\,dx$

Formule vi), où $a^2 = 4$

$$I = \sqrt{x^2+4} - 2\ln\left|\frac{2 + \sqrt{x^2+4}}{x}\right| + C$$

iii) $I = \int_3^5 x^2\sqrt{x^2 - 9}\,dx$

Formule ii), où $a^2 = 9$

$$\begin{aligned}
I &= \left(\frac{x}{8}(2x^2 - 9)\sqrt{x^2-9} - \frac{81}{8}\ln\left|x + \sqrt{x^2-9}\right|\right)\Bigg|_3^5\\
&= \frac{205}{2} - \frac{81}{3}\ln 3
\end{aligned}$$

iv) $I = \int_2^3\sqrt{(x^2-2)^3}\,dx$

Formule v), où $a^2 = 2$

$$\begin{aligned}
I &= \left(\frac{x}{8}(2x^2 - 10)\sqrt{x^2-2} + \frac{3}{2}\ln\left|x + \sqrt{x^2-2}\right|\right)\Bigg|_2^3\\
&= 3\sqrt{7} + \frac{3}{2}\ln\left(3 + \sqrt{7}\right) + \frac{\sqrt{2}}{2} - \frac{3}{2}\ln\left(2 + \sqrt{2}\right)
\end{aligned}$$

10. a) En posant $(4 - x^2)e^x = 0$

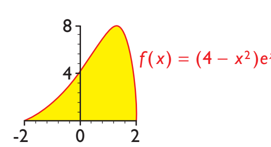
$$(2 - x)(2 + x)e^x = 0$$

nous obtenons $x = -2$ ou $x = 2$.

```
> f:=x->(4-x^2)*exp(x);
```

$$f := x \to (4 - x^2)e^x$$

```
> with(plots):
> y:=plot(f(x),x=-2..2,color=orange):
> c:=plot(f(x),x=-2..2,filled=true,color=yellow):
> display(y,c);
```



Calculons $I = \int (4 - x^2)e^x\,dx$

$$u = (4 - x^2) \qquad du = -2x\,dx \qquad\qquad dv = e^x\,dx \qquad v = e^x$$

$I = (4 - x^2)e^x + 2\int x\,e^x\,dx$

$u = x$, $du = dx$ | $dv = e^x\,dx$, $v = e^x$

$I = (4 - x^2)e^x + 2\left[xe^x - \int e^x\,dx\right]$

$I = (4 - x^2)e^x + 2xe^x - 2e^x + C$

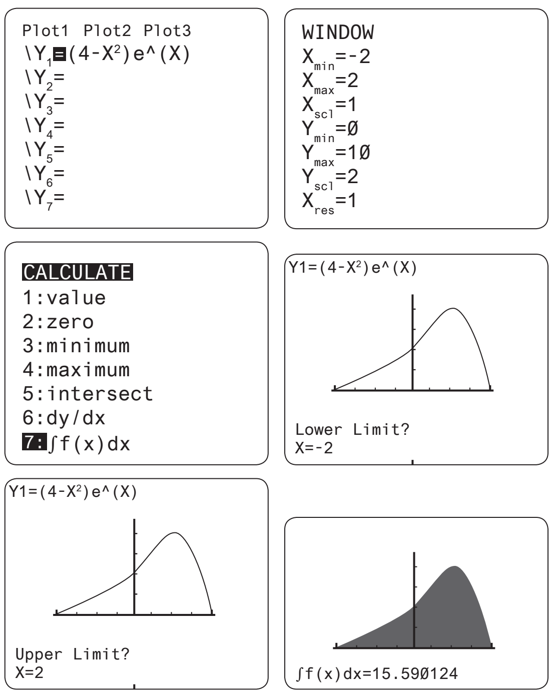
$$A = \int_{-2}^{2}(4 - x^2)e^x\,dx$$

$$= \left[(4 - x^2)e^x + 2xe^x - 2e^x\right]\Big|_{-2}^{2}$$

$$= [0 + 4e^2 - 2e^2] - [0 - 4e^{-2} - 2e^{-2}]$$

$$= \left(2e^2 + \frac{6}{e^2}\right)\text{u}^2$$

Vérification du résultat

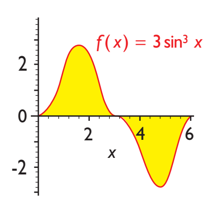


b) Soit $x \in [0, 2\pi]$.
En posant $3\sin^3 x = 0$
nous obtenons $\sin x = 0$
donc $x = 0$, $x = \pi$ ou $x = 2\pi$.

```
> f:=x->3*(sin(x))^3;
```

$$f := x \rightarrow 3\sin(x)^3$$

```
> with(plots):
> y:=plot(f(x),x=0..2*Pi,color=orange):
> c:=plot(f(x),x=0..2*Pi,filled=true,color=yellow):
> display(y,c);
```



Calculons $\int 3\sin^3 x\,dx = 3\int \sin^2 x \sin x\,dx$

$= 3\int (1 - \cos^2 x)\sin x\,dx$

$= -3\int (1 - u^2)\,du$

$= -3u + u^3 + C$

$= -3\cos x + \cos^3 x + C$

$u = \cos x$, $-du = \sin x\,dx$

$$A = \int_0^{\pi} 3\sin^3 x\,dx + \int_{\pi}^{2\pi}(0 - 3\sin^3 x)\,dx$$

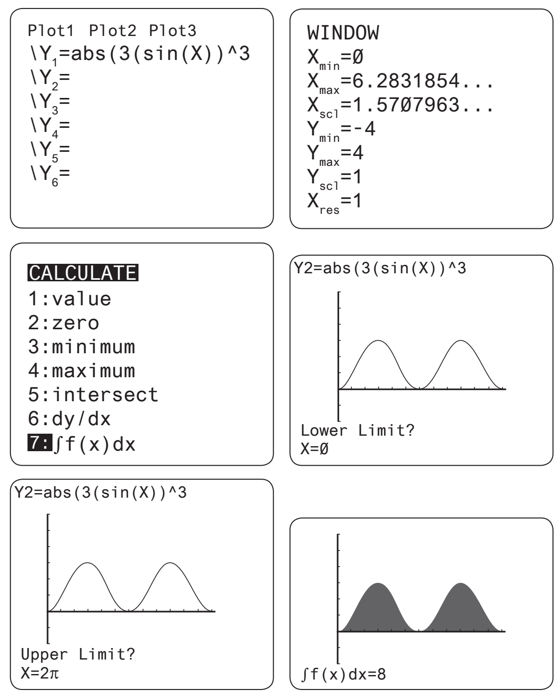
$$= (-3\cos x + \cos^3 x)\Big|_0^{\pi} + (3\cos x - \cos^3 x)\Big|_{\pi}^{2\pi}$$

$$= \left[(-3(-1) + (-1)) - (-3 + 1)\right] + \left[(3 - 1) - (3(-1) - (-1))\right]$$

$$= 4 + 4$$

$$= 8\text{ u}^2$$

Vérification du résultat



c) Soit $x \in [-1, 2]$.

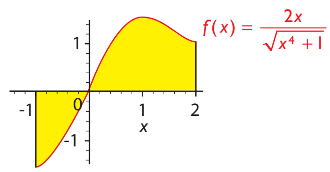
En posant $\dfrac{2x}{\sqrt{x^4 + 1}} = 0$

nous trouvons $x = 0$.

```
> f:=x->2*x/(x^4+1)^(1/2);
```

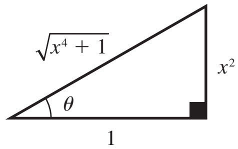
$$f := x \rightarrow 2\,\frac{x}{\sqrt{x^4 + 1}}$$

```
> with(plots):
> y:=plot(f(x),x=-1..2,color=orange):
> c:=plot(f(x),x=-1..2,filled=true,color=yellow):
> display(y,c);
```



Calculons $I = \int \frac{2x}{\sqrt{x^4+1}}\,dx$

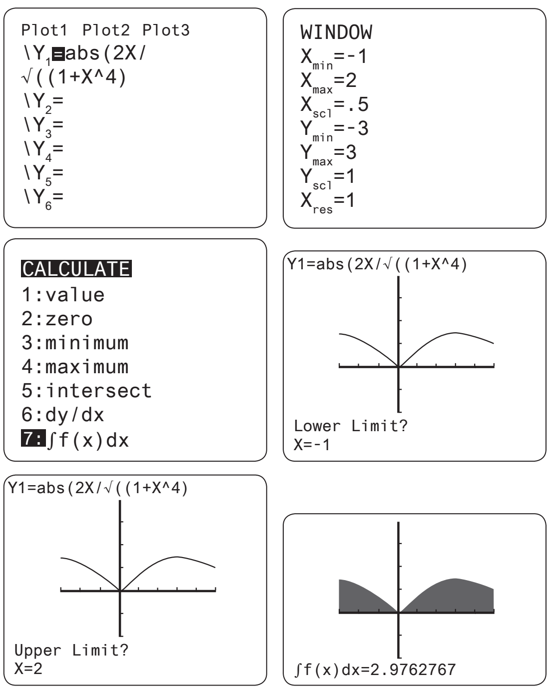
$$\begin{aligned} x^2 &= \tan\theta \\ 2x\,dx &= \sec^2\theta\,d\theta \\ \theta &= \text{Arc}\tan x^2 \end{aligned}$$



$$\begin{aligned} I &= \int \frac{\sec^2\theta}{\sqrt{\tan^2\theta+1}}\,d\theta \\ &= \int \sec\theta\,d\theta \\ &= \ln\left|\sec\theta+\tan\theta\right| + C \\ &= \ln\left|\sqrt{x^4+1}+x^2\right| + C \end{aligned}$$

$$\begin{aligned} A &= \int_{-1}^{0}\left(0-\frac{2x}{\sqrt{x^4+1}}\right)dx + \int_0^2 \frac{2x}{\sqrt{x^4+1}}\,dx \\ &= \left(-\ln\left|\sqrt{x^4+1}+x^2\right|\right)\Big|_{-1}^{0} + \left(\ln\left|\sqrt{x^4+1}+x^2\right|\right)\Big|_{0}^{2} \\ &= \left(0-\left(-\ln\left(1+\sqrt{2}\right)\right)\right) + \left(\ln\left(\sqrt{17}+4\right)-0\right) \\ &= \left(\ln\left(1+\sqrt{2}\right)+\ln\left(4+\sqrt{17}\right)\right)\text{u}^2 \end{aligned}$$

Vérification du résultat



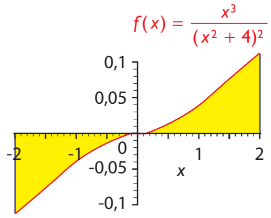
d) En posant $\frac{x^3}{(x^2+4)^2} = 0$

nous trouvons $x = 0$.

```
> f:=x->x^3/(x^2+4)^2;
```

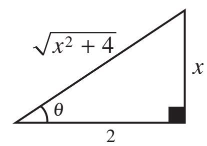
$$f := x \to \frac{x^3}{(x^2+4)^2}$$

```
> with(plots):
> y:=plot(f(x),x=-2..2,color=orange):
> c:=plot(f(x),x=-2..2,filled=true,color=yellow):
> display(y,c);
```



Calculons $I = \int \frac{x^3}{(x^2+4)^2}\,dx$

$$\begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \\ \theta &= \text{Arc}\tan\left(\frac{x}{2}\right) \end{aligned}$$

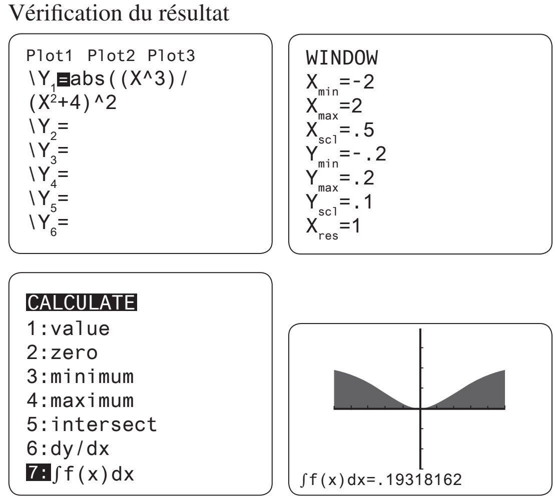


$$\begin{aligned} I &= \int \frac{8\tan^3\theta}{(4\tan^2\theta+4)^2}\,2\sec^2\theta\,d\theta \\ &= \frac{16}{16}\int \frac{\tan^3\theta}{\sec^4\theta}\sec^2\theta\,d\theta \\ &= \int \frac{\sin^3\theta}{\cos\theta}\,d\theta \\ &= \int \frac{(1-\cos^2\theta)}{\cos\theta}\sin\theta\,d\theta \\ &= -\int \frac{1}{u}\,du + \int u\,du \\ &= -\ln|u| + \frac{u^2}{2} + C \\ &= \frac{\cos^2\theta}{2} - \ln|\cos\theta| + C \\ &= \frac{2}{x^2+4} - \ln\left(\frac{2}{\sqrt{x^2+4}}\right) + C \end{aligned}$$

$$\begin{aligned} u &= \cos\theta \\ -du &= \sin\theta\,d\theta \end{aligned}$$

$$\begin{aligned} A &= 2\int_0^2 \frac{x^3}{(x^2+4)^2}\,dx \qquad \text{(par symétrie)} \\ &= 2\left(\frac{2}{x^2+4} - \ln\left(\frac{2}{\sqrt{x^2+4}}\right)\right)\Bigg|_0^2 \\ &= 2\left[\left(\frac{1}{4}-\ln\left(\frac{2}{\sqrt{8}}\right)\right) - \left(\frac{1}{2}-\ln(1)\right)\right] \\ &= \left(\ln 2 - \frac{1}{2}\right)\text{u}^2 \end{aligned}$$

Vérification du résultat



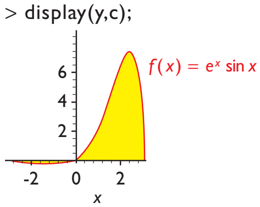
e) Soit $x \in [-\pi, \pi]$.

En posant $e^x \sin x = 0$
nous trouvons $x = -\pi$, $x = 0$ ou $x = \pi$.

```
> f:=x->exp(x)*sin(x);
```

$$f := x \to e^x \sin(x)$$

```
> with(plots):
> y:=plot(f(x),x=-Pi..Pi,color=orange):
> c:=plot(f(x),x=-Pi..Pi,filled=true,color=yellow):
> display(y,c);
```



Calculons $I = \int e^x \sin x\, dx$

$u = e^x$ $du = e^x\, dx$	$dv = \sin x\, dx$ $v = -\cos x$

$I = -e^x \cos x + \int e^x \cos x\, dx$

$u = e^x$ $du = e^x\, dx$	$dv = \cos x\, dx$ $v = \sin x$

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

$$2I = -e^x \cos x + e^x \sin x + C_1$$

$$I = \frac{e^x(\sin x - \cos x)}{2} + C$$

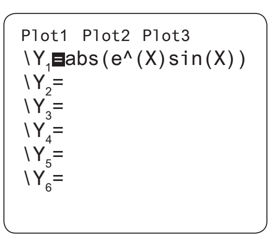
$$A = \int_{-\pi}^{0} \left(0 - e^x \sin x\right) dx + \int_0^{\pi} e^x \sin x\, dx$$

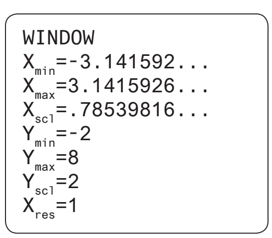
$$= \left(\frac{-e^x(\sin x - \cos x)}{2}\right)\Bigg|_{-\pi}^{0} + \left(\frac{e^x(\sin x - \cos x)}{2}\right)\Bigg|_0^{\pi}$$

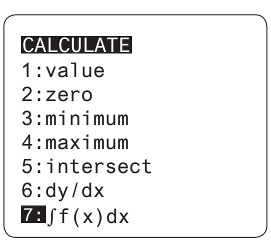
$$= \left(\frac{1}{2} + \frac{e^{-\pi}}{2}\right) + \left(\frac{1}{2} + \frac{e^{\pi}}{2}\right)$$

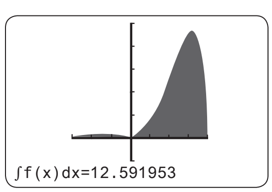
$$= \left(\frac{e^{\pi} + e^{-\pi} + 2}{2}\right) \text{u}^2$$

Vérification du résultat









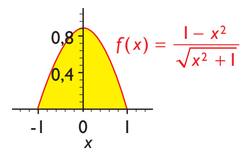
f) En posant $\dfrac{1 - x^2}{\sqrt{x^2 + 1}} = 0$

nous trouvons $x = -1$ ou $x = 1$.

```
> f:=x->(1-x^2)/(x^2+1)^(1/2);
```

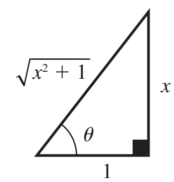
$$f := x \to \frac{1 - x^2}{\sqrt{x^2 + 1}}$$

```
> with(plots):
> y:=plot(f(x),x=-1..1,color=orange):
> c:=plot(f(x),x=-1..1,filled=true,color=yellow):
> display(y,c);
```



Calculons $\displaystyle\int \frac{1 - x^2}{\sqrt{x^2 + 1}}\, dx$

$x = \tan\theta$ $dx = \sec^2\theta\, d\theta$ $\theta = \text{Arc}\tan x$



$$\int \frac{1 - x^2}{\sqrt{x^2 + 1}}\, dx = \int \frac{1 - \tan^2\theta}{\sqrt{\tan^2\theta + 1}} \sec^2\theta\, d\theta$$

$$= \int (1 - \tan^2\theta) \sec\theta\, d\theta$$

$$= \int (\sec\theta - \tan^2\theta \sec\theta)\, d\theta$$

$$= \int \left(\sec\theta - (\sec^2\theta - 1)\sec\theta\right) d\theta$$

$$= \int 2\sec\theta\, d\theta - \int \sec^3\theta\, d\theta$$

$$= 2\ln\left|\sec\theta + \tan\theta\right| - \frac{1}{2}\left[\sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\right] + C$$

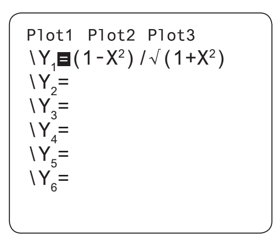
$$= \frac{3}{2}\ln\left|\sqrt{x^2 + 1} + x\right| - \frac{1}{2}x\sqrt{x^2 + 1} + C$$

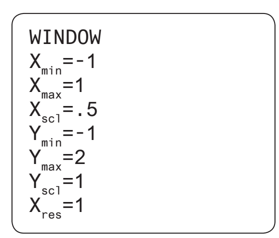
$$A = 2\int_0^1 \frac{1 - x^2}{\sqrt{x^2 + 1}}\, dx$$

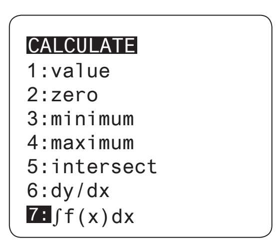
$$A = \left(3\ln\left|\sqrt{x^2 + 1} + x\right| - x\sqrt{x^2 + 1}\right)\Big|_0^1$$

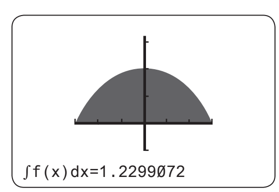
$$= \left(3\ln\left(1 + \sqrt{2}\right) - \sqrt{2}\right) \text{u}^2$$

Vérification du résultat









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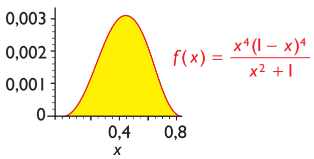
g) > f:=x->x^4*(1-x)^4/(1+x^2);

$$f := x \to \frac{x^4(1-x)^4}{x^2+1}$$

> with(plots):
> y:=plot(f(x),x=0..1,color=orange):
> c:=plot(f(x),x=0..1,filled=true,color=yellow):
> display(y,c);



$$\frac{x^4(1-x)^4}{x^2+1} = \frac{x^8 - 4x^7 + 6x^6 - 4x^5 + x^4}{x^2+1}$$
$$= x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}$$

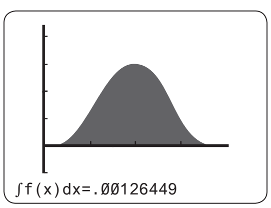
$$A = \int_0^1 \frac{x^4(1-x)^4}{x^2+1}\,dx$$
$$= \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right)dx$$
$$= \left(\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\,\text{Arc tan}\,x\right)\Bigg|_0^1$$
$$= \left(\frac{22}{7} - \pi\right)\text{u}^2$$

Vérification du résultat

```
Plot1 Plot2 Plot3
\Y1=((X^4)(1-X)^4)/
(1+X²)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=1
Xscl=.25
Ymin=-.002
Ymax=.004
Yscl=.001
Xres=1
```

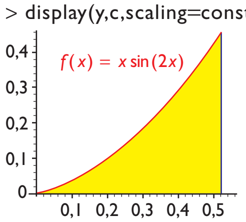
```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



h) > f:=x->x*sin(2*x);

$$f := x \to x\sin(2x)$$

> with(plots):
> y:=plot(f(x),x=0..Pi/6,color=orange):
> c:=plot(f(x),x=0..Pi/6,filled=true,color=yellow):
> display(y,c,scaling=constrained);



Calculons $I = \int x \sin 2x\,dx$

$u = x$ $du = dx$	$dv = \sin 2x\,dx$ $v = \frac{-\cos 2x}{2}$

$$I = \frac{-x\cos 2x}{2} + \frac{1}{2}\int \cos 2x\,dx$$
$$= \frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + C$$

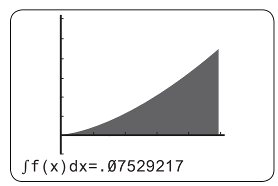
$$A = \int_0^{\frac{\pi}{6}} x\sin 2x\,dx$$
$$= \left(\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4}\right)\Bigg|_0^{\frac{\pi}{6}}$$
$$= \left(\frac{-\pi\cos\frac{\pi}{3}}{12} + \frac{\sin\frac{\pi}{3}}{4}\right) - 0$$
$$= \left(\frac{\sqrt{3}}{8} - \frac{\pi}{24}\right)\text{u}^2$$

Vérification du résultat

```
Plot1 Plot2 Plot3
\Y1=Xsin(2X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
Xmin=0
Xmax=.5
Xscl=.25
Ymin=-.15
Ymax=.5
Yscl=.1
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



11. a) $\frac{dP}{dt} = te^{\frac{t}{15}}$

$$\int dP = \int te^{\frac{t}{15}}\,dt$$

$u = t$ $du = dt$	$dv = e^{\frac{t}{15}}\,dt$ $v = 15e^{\frac{t}{15}}$

$$P = 15te^{\frac{t}{15}} - 15\int e^{\frac{t}{15}}\,dt$$
$$P = 15te^{\frac{t}{15}} - 225e^{\frac{t}{15}} + C$$

Si $t = 0$, $P = 20\,000$

$20\,000 = -225 + C$, donc $C = 20\,225$

d'où $P = 15te^{\frac{t}{15}} - 225e^{\frac{t}{15}} + 20\,225$

b) Si $t = 12$, $P \approx 20\,124$ habitants
si $t = 24$, $P \approx 20\,893$ habitants

12. a) $\frac{dP}{dt} = kP(75\,000 - P)$

b) $\displaystyle\int \frac{1}{P(75\,000 - P)}\,dP = \int k\,dt$

$$\frac{1}{P(75\,000 - P)} = \frac{A}{P} + \frac{B}{75\,000 - P}$$
$$= \frac{A(75\,000 - P) + BP}{P(75\,000 - P)}$$
$$1 = A(75\,000 - P) + BP$$

si $P = 0$, $1 = A(75\,000)$, donc $A = \dfrac{1}{75\,000}$

si $P = 75\,000$, $1 = B(75\,000)$, donc $B = \dfrac{1}{75\,000}$

$$\int \left[\frac{\frac{1}{75\,000}}{P} + \frac{\frac{1}{75\,000}}{75\,000 - P}\right] dP = \int k\,dt$$

$$\frac{1}{75\,000}\big[\ln|P| - \ln|75\,000 - P|\big] = kt + C$$

$$\frac{1}{75\,000}\ln\left(\frac{P}{75\,000 - P}\right) = kt + C$$

si $t = 0$, $P = 150$

Ainsi $\dfrac{1}{75\,000}\ln\left(\dfrac{150}{75\,000 - 150}\right) = C$

donc $C = \dfrac{1}{75\,000}\ln\left(\dfrac{1}{499}\right)$

Ainsi $\dfrac{1}{75\,000}\ln\left(\dfrac{P}{75\,000 - P}\right) = kt + \dfrac{1}{75\,000}\ln\left(\dfrac{1}{499}\right)$

$$\ln\left(\frac{P}{75\,000 - P}\right) = k_1 t + \ln\left(\frac{1}{499}\right)$$

si $t = 15$, $P = 1500$

$$\ln\left(\frac{1500}{75\,000 - 1500}\right) = k_1(15) + \ln\left(\frac{1}{499}\right)$$
$$\ln\left(\frac{1}{49}\right) = k_1(15) + \ln\left(\frac{1}{499}\right)$$
$$k_1 = \frac{\ln\left(\frac{1}{49}\right) - \ln\left(\frac{1}{499}\right)}{15}$$
$$k_1 = \frac{\ln\left(\frac{499}{49}\right)}{15}$$

$$\ln\left(\frac{P}{75\,000 - P}\right) = \frac{\ln\left(\frac{499}{49}\right)}{15}t + \ln\left(\frac{1}{499}\right) \quad \text{(équation 1)}$$

$$\frac{P}{75\,000 - P} = \frac{1}{499}e^{\frac{\ln\left(\frac{499}{49}\right)t}{15}}$$

En isolant P,

$$P = \frac{\frac{75\,000}{499}e^{\frac{\ln\left(\frac{499}{49}\right)t}{15}}}{1 + \frac{1}{499}e^{\frac{\ln\left(\frac{499}{49}\right)t}{15}}}$$

d'où $P = \dfrac{75\,000}{1 + 499e^{\frac{-\ln\left(\frac{499}{49}\right)t}{15}}}$

ou $P = \dfrac{75\,000}{1 + 499\left(\frac{49}{499}\right)^{\frac{t}{15}}}$ (équation 2)

c) Si $t = 30$ dans l'équation 2, $P \approx 12\,905$ habitants

d) Si $P = 37\,500$ dans l'équation 1,

$$\ln\left(\frac{37\,500}{75\,000 - 37\,500}\right) = \frac{\ln\left(\frac{499}{49}\right)}{15}t + \ln\left(\frac{1}{499}\right)$$
$$t = \frac{-15\ln\left(\frac{1}{499}\right)}{\ln\left(\frac{499}{49}\right)}$$
$$t = 40{,}1541\ldots$$

d'où $t \approx 40$ jours

13. a)
$$\frac{dP}{P(1 - P)} = k\,dt$$
$$\int \frac{1}{P(1 - P)}\,dP = \int k\,dt$$

$$\frac{1}{P(1 - P)} = \frac{A}{P} + \frac{B}{1 - P}$$
$$= \frac{A(1 - P) + BP}{P(1 - P)}$$
$$1 = A(1 - P) + BP$$

si $P = 0$, $1 = A$
si $P = 1$, $1 = B$

$$\int\left(\frac{1}{P} + \frac{1}{1 - P}\right)dP = \int k\,dt$$
$$\ln|P| - \ln|1 - P| = kt + C$$
$$\ln\left(\frac{P}{1 - P}\right) = kt + C$$

si $t = 0$, $P = 0{,}20$

$$\ln\left(\frac{0{,}20}{0{,}80}\right) = C, \text{ donc } C = \ln\left(\frac{1}{4}\right)$$
$$\ln\left(\frac{P}{1 - P}\right) = kt + \ln\left(\frac{1}{4}\right)$$

si $t = 1$, $P = 0{,}30$

$$\ln\left(\frac{0{,}30}{0{,}70}\right) = k + \ln\left(\frac{1}{4}\right), \text{ donc } k = \ln\left(\frac{12}{7}\right)$$

donc $\ln\left(\dfrac{P}{1 - P}\right) = \ln\left(\dfrac{12}{7}\right)t + \ln\left(\dfrac{1}{4}\right)$ (équation 1)

En posant $P = 0{,}40$ dans l'équation 1,

$$\ln\left(\frac{0{,}40}{0{,}60}\right) = \ln\left(\frac{12}{7}\right)t + \ln\left(\frac{1}{4}\right)$$
$$t = \frac{\ln\left(\frac{8}{3}\right)}{\ln\left(\frac{12}{7}\right)}$$

d'où $t \approx 1{,}82$ mois

4

b) En posant $t = 3$ dans l'équation 1,

$$\ln\left(\frac{P}{1-P}\right) = 3\ln\left(\frac{12}{7}\right) + \ln\left(\frac{1}{4}\right)$$

$$\ln\left(\frac{P}{1-P}\right) = \ln\left[\left(\frac{12}{7}\right)^3 \frac{1}{4}\right]$$

$$\frac{P}{1-P} = \frac{432}{343}$$

$$P = \frac{432}{343} - \frac{432}{343}P$$

$$P\left(1 + \frac{432}{343}\right) = \frac{432}{343}$$

$$P = \frac{432}{343} \cdot \frac{343}{775} \approx 0{,}5574$$

Elle peut espérer gagner ses élections, car $P \approx 55{,}74\,\%$ en sa faveur.

14.

	C.V.	I.P.	S.T.	F.P.
a)			X	
b)	X		X	
c)			X	
d)			X	
e)				X
f)			X	
g)	X		X	X
h)			X	X
i)	X			X
j)	X			
k)	X		X	X
l)	X			
m)		X		
n)		X		
o)	X			
p)		X		
q)	X			
r)		X		
s)		X		
t)	X			
u)		X		
v)	X			
w)	X			
x)	X			
y)		X		

Solutionnaire
Problèmes de synthèse

Chapitre 4 (page 257)

1. a) $I = \int x^5 e^{x^3}\,dx = \int x^3 x^2 e^{x^3}\,dx$

$u = x^3$, $du = 3x^2\,dx$; $dv = x^2 e^{x^3}\,dx$, $v = \dfrac{e^{x^3}}{3}$

$$I = \frac{x^3 e^{x^3}}{3} - \int x^2 e^{x^3}\,dx$$
$$= \frac{x^3 e^{x^3}}{3} - \frac{e^{x^3}}{3} + C$$

b) $I = \displaystyle\int \frac{\text{Arc tan}\sqrt{x}}{\sqrt{x}}\,dx$

$= 2\int \text{Arc tan}\, z\,dz \quad \left(z = \sqrt{x} \Rightarrow \dfrac{1}{\sqrt{x}}\,dx = 2\,dz\right)$

$u = \text{Arc tan}\, z$, $du = \dfrac{1}{1+z^2}\,dz$; $dv = dz$, $v = z$

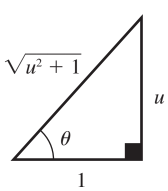
$$I = 2\left[z\,\text{Arc tan}\, z - \int \frac{z}{1+z^2}\,dz\right]$$
$$= 2\left[z\,\text{Arc tan}\, z - \frac{1}{2}\ln|1+z^2|\right] + C$$
$$= 2\sqrt{x}\,\text{Arc tan}\sqrt{x} - \ln|1+x| + C$$

c) $I = \int \tan^2\theta \tan\theta\sqrt{\sec\theta}\,d\theta$

$$= \int(\sec^2\theta - 1)\tan\theta\sqrt{\sec\theta}\,d\theta$$
$$= \int \sec^{\frac{5}{2}}\theta\tan\theta\,d\theta - \int \sec^{\frac{1}{2}}\theta\tan\theta\,d\theta$$
$$= \int \sec^{\frac{3}{2}}\theta\sec\theta\tan\theta\,d\theta - \int \sec^{\frac{-1}{2}}\theta\sec\theta\tan\theta\,d\theta$$
$$= \int u^{\frac{3}{2}}\,du - \int u^{\frac{-1}{2}}\,du \qquad (u = \sec\theta)$$
$$= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{1}{2}} + C$$
$$= \frac{2}{5}\sec^{\frac{5}{2}}\theta - 2\sqrt{\sec\theta} + C$$

d) $I = \int e^x\sqrt{1+e^{2x}}\,dx = \int\sqrt{1+u^2}\,du \qquad (u = e^x)$

$u = \tan\theta$, $du = \sec^2\theta\,d\theta$, $\theta = \text{Arc tan}\, u$



$$I = \int\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta$$
$$= \int\sec^3\theta\,d\theta$$
$$= \frac{\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|}{2} + C$$
$$= \frac{u\sqrt{u^2+1} + \ln\left|u + \sqrt{u^2+1}\right|}{2} + C$$
$$= \frac{e^x\sqrt{e^{2x}+1} + \ln\left(e^x + \sqrt{e^{2x}+1}\right)}{2} + C$$

e) $I = \displaystyle\int \frac{2}{1+\cos t}\,dt + \int\frac{\sin t}{1+\cos t}\,dt$

$$= 2\int\frac{1-\cos t}{\sin^2 t}\,dt - \int\frac{1}{u}\,du \qquad (u = 1+\cos t)$$
$$= 2\int\csc^2 t\,dt - 2\int\csc t\cot t\,dt - \int\frac{1}{u}\,du$$
$$= -2\cot t + 2\csc t - \ln|u| + C$$
$$= 2\csc t - 2\cot t - \ln(1+\cos t) + C$$

f) $I = \displaystyle\int\frac{1+\sin\theta}{2+\cos\theta}\,d\theta$

$u = \tan\left(\dfrac{\theta}{2}\right)$, $d\theta = \dfrac{2}{1+u^2}\,du$; $\sin\theta = \dfrac{2u}{1+u^2}$, $\cos\theta = \dfrac{1-u^2}{1+u^2}$

$$I = \int\frac{1+\dfrac{2u}{1+u^2}}{2+\dfrac{1-u^2}{1+u^2}}\left(\frac{2}{1+u^2}\right)du$$
$$= 2\int\frac{u^2+2u+1}{(3+u^2)(1+u^2)}\,du$$

$$\frac{u^2+2u+1}{(3+u^2)(1+u^2)} = \frac{Au+B}{3+u^2} + \frac{Cu+D}{1+u^2}$$

$u^2 + 2u + 1$
$= (Au+B)(1+u^2) + (Cu+D)(3+u^2)$
$= (A+C)u^3 + (B+D)u^2 + (A+3C)u + (B+3D)$

① $A + C = 0$
② $B + D = 1$
③ $A + 3C = 2$
④ $B + 3D = 1$

En résolvant, nous obtenons $A = -1$, $B = 1$, $C = 1$ et $D = 0$.

$$I = 2\int\left[\frac{-u+1}{3+u^2} + \frac{u}{1+u^2}\right]du$$
$$= -2\int \frac{u}{3+u^2}\,du + 2\int \frac{1}{3+u^2} + 2\int \frac{u}{1+u^2}\,du$$
$$= -2\left(\frac{\ln(3+u^2)}{2}\right) + \frac{2}{\sqrt{3}}\text{ Arc tan}\left(\frac{u}{\sqrt{3}}\right) + 2\left(\frac{\ln(1+u^2)}{2}\right) + C$$
$$= \ln\left(\frac{u^2+1}{u^2+3}\right) + \frac{2}{\sqrt{3}}\text{ Arc tan}\left(\frac{u}{\sqrt{3}}\right) + C$$
$$= \ln\left(\frac{\tan^2\left(\frac{\theta}{2}\right)+1}{\tan^2\left(\frac{\theta}{2}\right)+3}\right) + \frac{2}{\sqrt{3}}\text{ Arc tan}\left(\frac{\tan\left(\frac{\theta}{2}\right)}{\sqrt{3}}\right) + C$$

g) $I = \int \frac{1}{e^y + e^{-y}}\,dy = \int \frac{1}{e^{-y}(e^{2y}+1)}\,dy$
$$= \int \frac{e^y}{e^{2y}+1}\,dy$$
$$= \int \frac{1}{u^2+1}\,du \qquad (u = e^y)$$
$$= \text{Arc tan}\, u + C$$
$$= \text{Arc tan}\, e^y + C$$

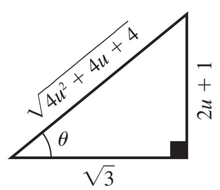
h) $I = \int \frac{e^x}{1-e^{3x}}\,dx = \int \frac{1}{1-u^3}\,du \qquad (u = 3^x)$
$$= \int \frac{1}{(1-u)(1+u+u^2)}\,du$$

En décomposant en une somme de fractions partielles,

$$\frac{1}{(1-u)(1+u+u^2)} = \frac{A}{1-u} + \frac{Bu+C}{u^2+u+1}$$
$$= \frac{\frac{1}{3}}{1-u} + \frac{\frac{1}{3}u + \frac{2}{3}}{u^2+u+1}$$

$$I = \frac{1}{3}\int \frac{1}{1-u}\,du + \frac{1}{3}\int \frac{u+2}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}\,du$$

$$u + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$$
$$du = \frac{\sqrt{3}}{2}\sec^2\theta\, d\theta$$
$$\theta = \text{Arc tan}\left(\frac{2u+1}{\sqrt{3}}\right)$$

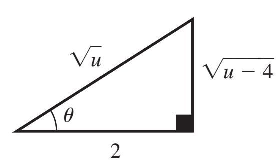


$$I = \frac{-1}{3}\ln|1-u| + \frac{1}{3}\int \frac{\left(\frac{\sqrt{3}}{2}\tan\theta + \frac{3}{2}\right)\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{3}{4}\tan^2\theta + \frac{3}{4}}\,d\theta$$
$$= \frac{-1}{3}\ln|1-e^x| + \frac{1}{3}\int \tan\theta\, d\theta + \frac{\sqrt{3}}{3}\int d\theta$$
$$= \frac{-1}{3}\ln|1-e^x| + \frac{1}{3}\ln|\sec\theta| + \frac{\sqrt{3}}{3}\theta + C_1$$
$$= \frac{-1}{3}\ln|1-e^x| + \frac{1}{3}\ln\left|\frac{\sqrt{4u^2+4u+4}}{3}\right| + \frac{\sqrt{3}}{3}\text{ Arc tan}\left(\frac{2u+1}{\sqrt{3}}\right) + C_1$$
$$= \frac{-1}{3}\ln|1-e^x| + \frac{1}{3}\ln\sqrt{u^2+u+1} + \frac{1}{3}\ln\frac{\sqrt{4}}{\sqrt{3}} + \frac{\sqrt{3}}{3}\text{ Arc tan}\left(\frac{2e^x+1}{\sqrt{3}}\right) + C_1$$
$$= \frac{-1}{3}\ln|1-e^x| + \frac{1}{3}\ln\sqrt{e^{2x}+e^x+1} + \frac{\sqrt{3}}{3}\text{ Arc tan}\left(\frac{2e^x+1}{\sqrt{3}}\right) + C$$

i) $I = \int \frac{\sin^2\theta\sin^2\theta}{\cos^2\theta}\,d\theta$
$$= \int \frac{\sin^2\theta\,(1-\cos^2\theta)}{\cos^2\theta}\,d\theta$$
$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta - \int \sin^2\theta\, d\theta$$
$$= \int \tan^2\theta\, d\theta - \int \sin^2\theta\, d\theta$$
$$= \int (\sec^2\theta - 1)\,d\theta - \left[\frac{-\sin\theta\cos\theta}{2} + \frac{1}{2}\int 1\, d\theta\right]$$
$$= \tan\theta - \theta + \frac{\sin\theta\cos\theta}{2} - \frac{\theta}{2} + C$$
$$= \tan\theta + \frac{\sin\theta\cos\theta}{2} - \frac{3\theta}{2} + C$$

j) $I = \int \frac{8}{u^2\sqrt{u-4}}\,du$

$$u = 4\sec^2\theta$$
$$du = 8\sec^2\theta\tan\theta\, d\theta$$
$$\theta = \text{Arc sec}\,\frac{\sqrt{u}}{2}$$



$$I = 8\int \frac{8\sec^2\theta\tan\theta}{16\sec^4\theta\,\sqrt{4\sec^2\theta - 4}}\,d\theta$$
$$= 2\int \cos^2\theta\, d\theta$$
$$= 2\left[\frac{\cos\theta\sin\theta + \theta}{2}\right] + C$$
$$= \frac{2}{\sqrt{u}}\,\frac{\sqrt{u-4}}{\sqrt{u}} + \text{Arc sec}\left(\frac{\sqrt{u}}{2}\right) + C$$
$$= \frac{2\sqrt{u-4}}{u} + \text{Arc sec}\left(\frac{\sqrt{u}}{2}\right) + C$$

k) $I = \int \sqrt{e^x - 1}\, dx$

Première façon :

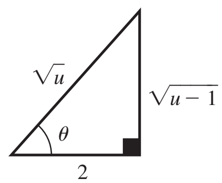
$$u = \sqrt{e^x - 1} \quad \Rightarrow \quad e^x = u^2 + 1$$
$$du = \frac{e^x}{2\sqrt{e^x - 1}}\, dx \quad \Rightarrow \quad du = \frac{u^2 + 1}{2u}\, dx$$

$$\begin{aligned}\int \sqrt{e^x - 1}\, dx &= 2\int \frac{u^2}{u^2 + 1}\, du \qquad \left(u = \sqrt{e^x - 1}\right)\\ &= 2\int\left(1 - \frac{1}{u^2 + 1}\right) du\\ &= 2u - 2 \text{ Arc tan } u + C\\ &= 2\sqrt{e^x - 1} - 2 \text{ Arc tan } \sqrt{e^x - 1} + C\end{aligned}$$

Deuxième façon :

$$I = \int \frac{e^x\sqrt{e^x - 1}}{e^x}\, dx = \int \frac{\sqrt{u - 1}}{u}\, du \qquad (u = e^x)$$

$$u = \sec^2 \theta$$
$$du = 2\sec^2 \theta \tan \theta\, d\theta$$
$$\theta = \text{Arc sec } \sqrt{u}$$



$$\begin{aligned}I &= \int \frac{\sqrt{\sec^2 \theta - 1}}{\sec^2 \theta}\, 2\sec^2 \theta \tan \theta\, d\theta\\ &= 2\int \tan^2 \theta\, d\theta\\ &= 2\int (\sec^2 \theta - 1)\, d\theta\\ &= 2\tan \theta - 2\theta + C\\ &= 2\sqrt{u - 1} - 2 \text{ Arc sec } \sqrt{u} + C\\ &= 2\sqrt{e^x - 1} - 2 \text{ Arc sec } \sqrt{e^x} + C\end{aligned}$$

l) $I = \int \frac{2x \ln x}{(1 + x^2)^2}\, dx$

$$u = \ln x \qquad du = \frac{1}{x}\, dx$$

$$dv = \frac{2x}{(1 + x^2)^2}\, dx \qquad v = \frac{-1}{(1 + x^2)}$$

$$I = \frac{-\ln x}{1 + x^2} + \int \frac{1}{x(1 + x^2)}\, dx$$

$$\begin{aligned}\frac{1}{x(1 + x^2)} &= \frac{A}{x} + \frac{Bx + C}{1 + x^2}\\ &= \frac{1}{x} + \frac{-x}{1 + x^2}\end{aligned}$$

$$\begin{aligned}I &= \frac{-\ln x}{1 + x^2} + \int\left(\frac{1}{x} - \frac{x}{1 + x^2}\right) dx\\ &= \frac{-\ln x}{1 + x^2} + \ln|x| - \frac{\ln(1 + x^2)}{2} + C\\ &= \frac{x^2 \ln x}{1 + x^2} - \frac{\ln(1 + x^2)}{2} + C \qquad (\text{car } x > 0)\end{aligned}$$

m) $I = \int \sqrt{4 + 2\sqrt{x}}\, dx$

$$u = 4 + 2\sqrt{x} \Rightarrow \sqrt{x} = \frac{u - 4}{2}$$
$$du = \frac{1}{\sqrt{x}}\, dx \quad \Rightarrow \quad dx = \left(\frac{u - 4}{2}\right) du$$

$$\begin{aligned}I &= \int \sqrt{u}\left(\frac{u - 4}{2}\right) du\\ &= \frac{1}{2}\int u^{\frac{3}{2}}\, du - 2\int u^{\frac{1}{2}}\, du\\ &= \frac{1}{2}\frac{u^{\frac{5}{2}}}{\frac{5}{2}} - 2\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C\\ &= \frac{\sqrt{\left(4 + 2\sqrt{x}\right)^5}}{5} - \frac{4\sqrt{\left(4 + 2\sqrt{x}\right)^3}}{3} + C\end{aligned}$$

n) $I = \int (\tan^2 x - \tan x)e^{-x}\, dx$

Calculons d'abord $\int \tan x e^{-x}\, dx$.

$$u = \tan x \qquad du = \sec^2 x\, dx$$

$$dv = e^{-x}\, dx \qquad v = -e^{-x}$$

$$\begin{aligned}\int \tan x e^{-x}\, dx &= -e^{-x} \tan x + \int (\sec^2 x)e^{-x}\, dx\\ &= -e^{-x} \tan x + \int (\tan^2 x + 1)e^{-x}\, dx\\ &= -e^{-x} \tan x + \int \tan^2 x e^{-x}\, dx + \int e^{-x}\, dx\end{aligned}$$

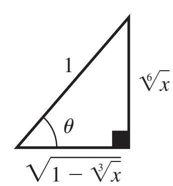
$$\int \tan^2 x e^{-x}\, dx - \int \tan x e^{-x}\, dx = e^{-x} \tan x - \int e^{-x}\, dx$$

$$I = e^{-x} \tan x + e^{-x} + C$$

2. a) $I = \int \frac{1}{\sqrt{x}\left(1 - \sqrt[3]{x}\right)}\, dx$

Première façon : substitution trigonométrique

$$\sqrt[3]{x} = \sin^2 \theta$$
$$x = \sin^6 \theta$$
$$dx = 6\sin^5 \theta \cos \theta\, d\theta$$
$$\theta = \text{Arc sin}\left(\sqrt[6]{x}\right)$$



$$\begin{aligned}I &= \int \frac{6\sin^5 \theta \cos \theta}{\sin^3 \theta\,(1 - \sin^2 \theta)}\, d\theta\\ &= 6\int \frac{\sin^2 \theta \cos \theta}{\cos^2 \theta}\, d\theta\\ &= 6\int \frac{\sin^2 \theta}{\cos \theta}\, d\theta\\ &= 6\int \frac{1 - \cos^2 \theta}{\cos \theta}\, d\theta\\ &= 6\int (\sec \theta - \cos \theta)\, d\theta\\ &= 6\Big[\ln|\sec \theta + \tan \theta| - \sin \theta\Big] + C_1\\ &= 6\left[\ln\left|\frac{1}{\sqrt{1 - \sqrt[3]{x}}} + \frac{\sqrt[6]{x}}{\sqrt{1 - \sqrt[3]{x}}}\right| - \sqrt[6]{x}\right] + C_1\\ &= 6\ln\left|1 + \sqrt[6]{x}\right| - 6\ln\left|\sqrt{1 - \sqrt[3]{x}}\right| - \sqrt[6]{x} + C_1\\ &= 6\ln\left|1 + \sqrt[6]{x}\right| - 3\ln\left|1 - \sqrt[3]{x}\right| - 6\sqrt[6]{x} + C_1\end{aligned}$$

Deuxième façon : changement de variable

$$I = \int \frac{6u^5}{u^3(1-u^2)}\,du \qquad \left(u = x^{\frac{1}{6}}\right)$$

$$= 6\int \frac{u^2}{1-u^2}\,du$$

En décomposant en une somme de fractions partielles,

$$\frac{u^2}{1-u^2} = -1 + \frac{A}{1-u} + \frac{B}{1+u}$$

$$= -1 + \frac{\frac{1}{2}}{1-u} + \frac{\frac{1}{2}}{1+u}$$

$$I = 6\left[\int (-1)\,du + \frac{1}{2}\int \frac{1}{1-u}\,du + \frac{1}{2}\int \frac{1}{1+u}\,du\right]$$

$$= 6\left[-u - \frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u|\right] + C_2$$

$$= 3\ln\left|1+\sqrt[6]{x}\right| - 3\ln\left|1-\sqrt[6]{x}\right| - 6\sqrt[6]{x} + C_2$$

$$= 3\ln\left|\frac{1+\sqrt[6]{x}}{1-\sqrt[6]{x}}\right| - 6\sqrt[6]{x} + C_2$$

b) $I = \displaystyle\int \frac{\cos x}{\sin x\sqrt{1+\sin x}}\,dx$

Première façon : changement de variable

$$I = 2\int \frac{1}{u^2-1}\,du \qquad \left(u = \sqrt{1+\sin x}\right)$$

$$= 2\int\left[\frac{\frac{1}{2}}{u-1} + \frac{\frac{-1}{2}}{u+1}\right]du$$

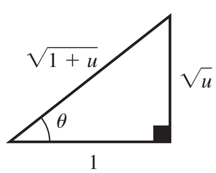
$$= \ln|u-1| - \ln|u+1| + C_1$$

$$= \ln\left|\frac{\sqrt{1+\sin x}-1}{\sqrt{1+\sin x}+1}\right| + C_1$$

Deuxième façon : substitution trigonométrique

$$I = \int \frac{1}{u\sqrt{1+u}}\,du \qquad (u = \sin x)$$

$u = \tan^2\theta$
$du = 2\tan\theta\sec^2\theta\,d\theta$
$\theta = \text{Arc}\tan\sqrt{u}$



$$I = \int \frac{2\tan\theta\sec^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}\,d\theta$$

$$= 2\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= 2\int \csc\theta\,d\theta$$

$$= 2\ln|\csc\theta - \cot\theta| + C_2$$

$$= 2\ln\left|\frac{\sqrt{1+u}}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right| + C_2$$

$$= 2\ln\left|\frac{\sqrt{1+\sin x}-1}{\sqrt{\sin x}}\right| + C_2$$

3. a) $(x^2 - x)\dfrac{dy}{dx} = y^2 + y$

$$\int \frac{1}{y(y+1)}\,dy = \int \frac{1}{x(x-1)}\,dx$$

En décomposant $\dfrac{1}{y(y+1)}$ et $\dfrac{1}{x(x-1)}$ en somme de fractions partielles, nous obtenons

$$\int\left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \int\left(\frac{-1}{x} + \frac{1}{x-1}\right)dx$$

$$\ln|y| - \ln|y+1| = -\ln|x| + \ln|x-1| + C$$

$$\ln\left|\frac{y}{y+1}\right| = \ln\left|\frac{x-1}{x}\right| + C$$

i) En remplaçant y par 2 et x par 3,

$$\ln\left(\frac{2}{3}\right) = \ln\left(\frac{2}{3}\right) + C, \text{ donc } C = 0$$

ainsi $\ln\left|\dfrac{y}{y+1}\right| = \ln\left|\dfrac{x-1}{x}\right|$

$$\frac{y}{y+1} = \frac{x-1}{x}$$

$$yx = yx - y + x - 1$$

d'où $y = x - 1$

ii) En remplaçant y par 2 et x par $\dfrac{3}{5}$,

$$\ln\left(\frac{2}{3}\right) = \ln\left|\frac{-2}{3}\right| + C, \text{ donc } C = 0$$

ainsi $\dfrac{y}{y+1} = -\left(\dfrac{x-1}{x}\right)$

$$yx = -yx + y - x + 1$$

d'où $y = \dfrac{1-x}{2x-1}$

b) $y\,dy = e^{2x}e^{y}\sin(5\pi e^{2x})\,dx$

$$\int ye^{-y}\,dy = \int e^{2x}\sin(5\pi e^{2x})\,dx$$

$u = y \qquad dv = e^{-y}dy$
$du = dy \qquad v = -e^{-y}$

$z = 5\pi e^{2x}$
$dz = 10\pi e^{2x}\,dx$

$$-ye^{-y} + \int e^{-y}\,dy = \frac{1}{10\pi}\int \sin z\,dz$$

$$-ye^{-y} - e^{-y} = \frac{-\cos z}{10\pi} + C$$

$$ye^{-y} + e^{-y} = \frac{\cos(5\pi e^{2x})}{10\pi} + C_1$$

En remplaçant y par 0 et x par 0, nous obtenons

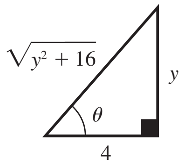
$$1 = \frac{-1}{10\pi} + C_1, \text{ donc } C_1 = 1 + \frac{1}{10\pi}$$

d'où $ye^{-y} + e^{-y} = \dfrac{\cos(5\pi e^{2x})}{10\pi} + 1 + \dfrac{1}{10\pi}$

c) $$\int \frac{x}{\sqrt{25-x^2}}\,dx = \int \frac{1}{\sqrt{y^2+16}}\,dy$$

$u = 25 - x^2$
$du = -2x\,dx$

$y = 4\tan\theta$
$dy = 4\sec^2\theta\,d\theta$



$$\frac{-1}{2}\int \frac{1}{\sqrt{u}}\,du = \int \sec\theta\,d\theta$$
$$-\sqrt{u} = \ln|\sec\theta + \tan\theta| + C$$
$$-\sqrt{25-x^2} = \ln\left|\frac{\sqrt{y^2+16}}{4} + \frac{y}{4}\right| + C$$
$$-\sqrt{25-x^2} = \ln\left(\sqrt{y^2+16} + y\right) + C_1$$

En remplaçant y par 3 et x par -4, nous obtenons

$-\sqrt{9} = \ln\left(\sqrt{25}+3\right) + C_1$, donc $C_1 = -3 - \ln 8$

d'où $-\sqrt{25-x^2} = \ln\left(\sqrt{y^2+16}+y\right) - 3 - \ln 8$

4. a) Puisque

$$\int \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

alors

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{-\sin^{n-1}x\cos x}{n}\Bigg|_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx$$
$$= 0 + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx, \text{ pour } n \geq 2$$

d'où $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}x\,dx$, pour $n \geq 2$

b) $$\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \frac{6}{7}\int_0^{\frac{\pi}{2}} \sin^5 x\,dx$$
$$= \frac{6}{7}\times\frac{4}{5}\int_0^{\frac{\pi}{2}} \sin^3 x\,dx$$
$$= \frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin x\,dx$$
$$= \frac{16}{35}(-\cos x)\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{16}{35}\left[-\cos\left(\frac{\pi}{2}\right) - (-\cos 0)\right]$$
$$= \frac{16}{35}$$

c) $$\int_0^{\frac{\pi}{2}} \sin^{20}x\,dx = \frac{19}{20}\int_0^{\frac{\pi}{2}} \sin^{18}x\,dx$$
$$= \frac{19}{20}\times\frac{17}{18}\int_0^{\frac{\pi}{2}} \sin^{16}x\,dx$$
$$= \frac{19}{20}\times\frac{17}{18}\times\frac{15}{16}\int_0^{\frac{\pi}{2}} \sin^{14}x\,dx$$
$$\vdots$$
$$= \left(\frac{19}{20}\right)\left(\frac{17}{18}\right)\left(\frac{15}{16}\right)\left(\frac{13}{14}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\int_0^{\frac{\pi}{2}} \sin^0 x\,dx$$
$$= \left(\frac{19}{20}\right)\left(\frac{17}{18}\right)\left(\frac{15}{16}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left[x\Big|_0^{\frac{\pi}{2}}\right]$$
$$= \left(\frac{19}{20}\right)\left(\frac{17}{18}\right)\left(\frac{15}{16}\right)\cdots\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)$$

5. a) Trouvons les points d'intersection des courbes.

$$\frac{37x^2}{(x-6)(x^2+1)} = \frac{37}{(x-7)}$$
$$x^2(x-7) = (x-6)(x^2+1)$$
$$x^3 - 7x^2 = x^3 - 6x^2 + x - 6$$
$$0 = x^2 + x - 6$$
$$0 = (x+3)(x-2)$$

donc $x = -3$ ou $x = 2$

Les points d'intersection sont (-3 ; -3,7) et (2 ; -7,4).

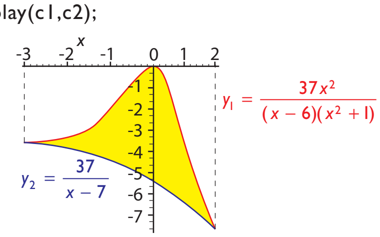
```
> f:=x->37*x^2/((x-6)*(x^2+1));
```

$$f := x \rightarrow 37\,\frac{x^2}{(x-6)(x^2+1)}$$

```
> g:=x->37/(x-7);
```

$$g := x \rightarrow 37\,\frac{1}{x-7}$$

```
> with(plots):
> c1:=plot([f(x),g(x)],x=-3..2,color=[red,blue]):
> c2:=plot([max(min(f(x),g(x)),0),min(max(f(x),g(x)),0),
  f(x),g(x)],x=-3..2,filled=true,color=[white,white,
  yellow,yellow]):
> display(c1,c2);
```



$$A = \int_{-3}^{2}\left(\frac{37x^2}{(x-6)(x^2+1)} - \frac{37}{x-7}\right)dx$$

Décomposons $\dfrac{37x^2}{(x-6)(x^2+1)}$ en une somme de fractions partielles.

$$\frac{37x^2}{(x-6)(x^2+1)} = \frac{A}{x-6} + \frac{Bx+C}{x^2+1}$$
$$= \frac{36}{x-6} + \frac{x+6}{x^2+1}$$

4

donc $\int_{-3}^{2}\left(\frac{37x^2}{(x-6)(x^2+1)} - \frac{37}{x-7}\right)dx$

$= \int_{-3}^{2}\left(\frac{36}{x-6} + \frac{x}{x^2+1} + \frac{6}{x^2+1} - \frac{37}{x-7}\right)dx$

$= \left(36\ln|x-6| + \frac{1}{2}\ln|x^2+1| + 6\,\text{Arc}\tan x - 37\ln|x-7|\right)\Big|_{-3}^{2}$

$= 36\ln 4 + \frac{1}{2}\ln 5 + 6\,\text{Arc}\tan 2 - 37\ln 5 - 36\ln 9 - \frac{1}{2}\ln 10 - 6\,\text{Arc}\tan(-3) + 37\ln 10$

$= 36\ln\left(\frac{4}{9}\right) + \frac{73}{2}\ln 2 + 6\left(\text{Arc}\tan 2 - \text{Arc}\tan(-3)\right)$

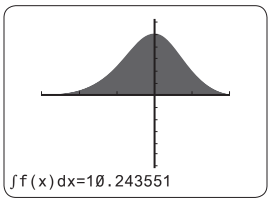
$\approx 10{,}24\ \text{u}^2$

Vérification du résultat

```
Plot1  Plot2  Plot3
\Y1=37/(X-7)
\Y2=37X²/((X-6)
(1+X²)
\Y3■(37X²/((X-6)
(1+X²))-37/(X-7)
\Y4=
```

```
WINDOW
Xmin=-3
Xmax=2
Xscl=1
Ymin=-8
Ymax=6
Yscl=1
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



b) Trouvons l'équation de la tangente.

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$, ainsi $f'(1) = 1$

donc l'équation de la tangente est

$y = ax + b$

$y = 1x + b \quad (\text{car } f'(1) = a = 1)$

La droite passe par le point $(1,\ f(1))$, c'est-à-dire $(1, 0)$

$0 = 1(1) + b$

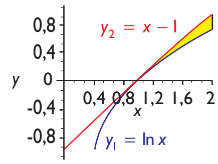
$-1 = b$

ainsi $y = x - 1$

```
> f:=x->ln(x);
```
$f := x \to \ln(x)$

```
> g:=x->x-1;
```
$g := x \to x - 1$

```
> with(plots):
> c1:=plot([f(x),g(x)],x=0..2,y=-1..1,color=[red,blue]):
> c2:=plot([max(min(f(x),g(x)),0),min(max(f(x),g(x)),0),
  f(x),g(x)],x=1..2,filled=true,color=[white,white,
  yellow,yellow]):
> display(c1,c2,scaling=constrained);
```



$A = \int_1^2 ((x-1) - \ln x)\,dx$

$= \left(\frac{x^2}{2} - x - (x\ln x - x)\right)\Big|_1^2$

$= \left(\frac{4}{2} - 2\ln 2\right) - \left(\frac{1}{2} - 1\ln 1\right)$

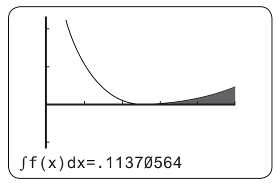
$= \left(\frac{3}{2} - 2\ln 2\right)\text{u}^2 \approx 0{,}11\ \text{u}^2$

Vérification du résultat

```
Plot1  Plot2  Plot3
\Y1■X-1-ln(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
Xmin=0
Xmax=2
Xscl=.4
Ymin=-1
Ymax=1
Yscl=.4
Xres=1
```

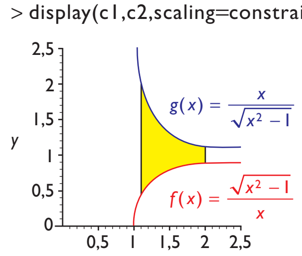
```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



c)
```
> f:=x->(x^2-1)^(1/2)/x;
```
$f := x \to \frac{\sqrt{x^2-1}}{x}$

```
> g:=x->x/(x^2-1)^(1/2);
```
$g := x \to \frac{x}{\sqrt{x^2-1}}$

```
> with(plots):
> c1:=plot([f(x),g(x)],x=0..2.5,y=0..2.5,color=[red,blue]):
> c2:=plot([max(min(f(x),g(x)),0),min(max(f(x),g(x)),0),
  f(x),g(x)],x=2/(3)^(1/2)..2,filled=true,color=
  [white,white,yellow,yellow]):
> display(c1,c2,scaling=constrained);
```



4

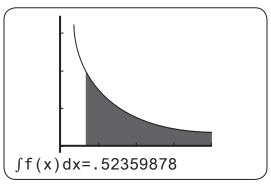
$$A = \int_{\frac{2}{\sqrt{3}}}^{2} \left[\frac{x}{\sqrt{x^2-1}} - \frac{\sqrt{x^2-1}}{x}\right] dx$$
$$= \int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$$
$$= \text{Arc sec } x \Big|_{\frac{2}{\sqrt{3}}}^{2}$$
$$= \text{Arc sec } 2 - \text{Arc sec}\left(\frac{2}{\sqrt{3}}\right)$$
$$= \frac{\pi}{3} - \frac{\pi}{6}$$
$$= \frac{\pi}{6}\ \text{u}^2$$

Vérification du résultat

```
Plot1 Plot2 Plot3
\Y1=X/(√(X²-1))
-(√(X²-1))/X
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=1
Xmax=2
Xscl=.25
Ymin=0
Ymax=2.5
Yscl=.25
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



d) Trouvons les points d'intersection des courbes.

$$(\tan^2 x - \tan x)e^{-x} = 2e^{-x}$$
$$\tan^2 x - \tan x - 2 = 0$$
$$(\tan x - 2)(\tan x + 1) = 0$$

$\tan x = 2$ si $x \approx 1{,}107\ldots \pm k\pi$, à rejeter

$\tan x = -1$ si $x = \dfrac{3\pi}{4}$

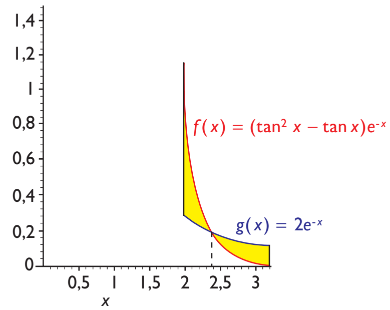
```
> f:=x->((tan(x)^2-tan(x))*exp(-x));
```

$$f := x \to \left(\tan(x)^2 - \tan(x)\right)e^{(-x)}$$

```
> g:=x->2*exp(-x);
```

$$g := x \to 2e^{(-x)}$$

```
> with(plots):
> y1:=plot(f(x),x=5*Pi/8..Pi,color=orange):
> y2:=plot(g(x),x=5*Pi/8..Pi,color=blue):
> c1:=plot(g(x),x=5*Pi/8..3*Pi/4,filled=true,color=white):
> c2:=plot(f(x),x=5*Pi/8..3*Pi/4,filled=true,color=yellow):
> c3:=plot(f(x),x=3*Pi/4..Pi,filled=true,color=white):
> c4:=plot(g(x),x=3*Pi/4..Pi,filled=true,color=yellow):
> x1:=plot([[5*Pi/8,g(5*Pi/8)],[5*Pi/8,f(5*Pi/8)]],
  color=black):
> x2:=plot([[Pi,f(Pi)],[Pi,g(Pi)]],color=black):
> x3:=plot([[3*Pi/4,0],[3*Pi/4,f(3*Pi/4)]],color=black,
  linestyle=4):
> display(y1,y2,c1,c2,c3,c4,x1,x2,x3,
  view=[0..Pi+0.2,0..1.5]);
```



$$A = A_1 + A_2$$
$$= \int_{\frac{5\pi}{8}}^{\frac{3\pi}{4}} \left[\left(\tan^2 x - \tan x\right)e^{-x} - 2e^{-x}\right]dx + \int_{\frac{3\pi}{4}}^{\pi} \left[2e^{-x} - \left(\tan^2 x - \tan x\right)e^{-x}\right]dx$$
$$= \left[e^{-x}\tan x + e^{-x} + 2e^{-x}\right]\Big|_{\frac{5\pi}{8}}^{\frac{3\pi}{4}} + \left[-2e^{-x} - (e^{-x}\tan x + e^{-x}\right]\Big|_{\frac{3\pi}{4}}^{\pi}$$

(voir le numéro 1 n), page 125)

$$= e^{-x}(\tan x + 3)\Big|_{\frac{5\pi}{8}}^{\frac{3\pi}{4}} + \left[-e^{-x}(\tan x + 3)\right]\Big|_{\frac{3\pi}{4}}^{\pi}$$
$$\approx 0{,}107\,33\ldots + 0{,}059\,91\ldots$$
$$\approx 0{,}167\,25\ldots$$

d'où $A \approx 0{,}167$ u²

6. a) Déterminons les points d'intersection du cercle et de l'ellipse.

De ① $x^2 + y^2 = 25$, nous avons $x^2 = 25 - y^2$.

En substituant dans ② $\dfrac{x^2}{32} + \dfrac{y^2}{18} = 1$, nous obtenons

$$\frac{25 - y^2}{32} + \frac{y^2}{18} = 1$$
$$\frac{14y^2}{576} = 1 - \frac{25}{32}$$
$$y^2 = 9,\ \text{ainsi } y = \pm 3$$

En posant $y = 3$ et $y = -3$ dans ①, nous obtenons $x = \pm 4$ et les points d'intersection sont A(-4, 3), B(4, 3), C(4, -3) et D(-4, -3).

$$A_1 = \int_{-4}^{4}\left[\sqrt{25 - x^2} - \sqrt{18 - \frac{9}{16}x^2}\right]dx$$
$$= 2\int_{0}^{4}\left[\sqrt{25 - x^2} - \frac{3}{4}\sqrt{(32 - x^2)}\right]dx$$

En utilisant les formules provenant des tables d'intégration, nous obtenons

$$\int \sqrt{25 - x^2}\,dx = \frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\text{Arc sin}\left(\frac{x}{5}\right) \text{ et}$$

$$\int \frac{3}{4}\sqrt{32 - x^2}\,dx = \frac{3}{4}\left[\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\text{Arc sin}\left(\frac{x}{\sqrt{32}}\right)\right]$$

$$A_1 = 2\left\{\left[\frac{4}{2}\sqrt{9} + \frac{25}{2}\,\text{Arc sin}\left(\frac{4}{5}\right) - 0\right] - \frac{3}{4}\left[\frac{4}{2}\sqrt{16} + \frac{32}{2}\,\text{Arc sin}\left(\frac{4}{\sqrt{32}}\right) - 0\right]\right\}$$

$$= 25\,\text{Arc sin}\left(\frac{4}{5}\right) - 24\,\text{Arc sin}\left(\frac{4}{\sqrt{32}}\right)$$

d'où $A_1 \approx 4{,}33\ \text{u}^2$

$$A_2 = \int_{-3}^{3}\left[\sqrt{32 - \frac{32}{18}y^2} - \sqrt{25 - y^2}\right]dy$$

$$= 2\int_{0}^{3}\left[\frac{4}{3}\sqrt{18 - y^2} - \sqrt{25 - y^2}\right]dy$$

En utilisant les formules d'intégrales, nous obtenons

$$\int \frac{4}{3}\sqrt{18 - y^2}\,dy = \frac{4}{3}\left[\frac{y}{2}\sqrt{18 - y^2} + \frac{18}{2}\,\text{Arc sin}\left(\frac{y}{\sqrt{18}}\right)\right]$$

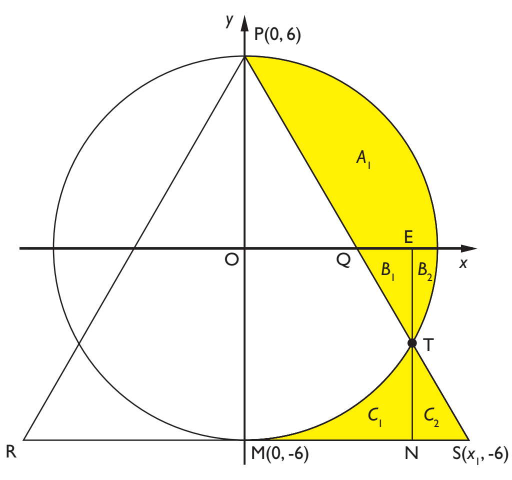
et $\int \sqrt{25 - y^2}\,dy = \frac{y}{2}\sqrt{25 - y^2} + \frac{25}{2}\,\text{Arc sin}\left(\frac{y}{5}\right)$

$$A_2 = 2\left\{\frac{4}{3}\left[\frac{3}{2}\sqrt{9} + 9\,\text{Arc sin}\left(\frac{3}{\sqrt{18}}\right) - 0\right] - \left[\frac{3}{2}\sqrt{16} + \frac{25}{2}\,\text{Arc sin}\left(\frac{3}{5}\right) - 0\right]\right\}$$

$$= 24\,\text{Arc sin}\left(\frac{3}{\sqrt{18}}\right) - 25\,\text{Arc sin}\left(\frac{3}{5}\right)$$

d'où $A_2 \approx 2{,}76\ \text{u}^2$

b) Déterminons d'abord les coordonnées des points S, Q et T.



$$\left(\overline{\text{PM}}\right)^2 + \left(\overline{\text{MS}}\right)^2 = \left(\overline{\text{PS}}\right)^2$$

$$(12)^2 + x_1^2 = (2x_1)^2$$

$$144 = 3x_1^2,\ \text{donc } x_1 = 4\sqrt{3}$$

$$\frac{y - 6}{x - 0} = \frac{-12}{4\sqrt{3}},\ \text{ainsi } \frac{y - 6}{x} = -\sqrt{3}$$

donc $y = -\sqrt{3}x + 6$ est l'équation de la droite passant par les points P(0, 6) et $S(4\sqrt{3},\ -6)$.

En posant $y = 0$, nous trouvons $0 = -\sqrt{3}x + 6$,
donc $x = 2\sqrt{3}$,
ainsi les coordonnées du point Q sont $Q(2\sqrt{3}, 0)$.

En remplaçant y par $(-\sqrt{3}x + 6)$ dans l'équation du cercle $x^2 + y^2 = 36$, nous obtenons

$$x^2 + (-\sqrt{3}x + 6)^2 = 36$$

$$x^2 + 3x^2 - 12\sqrt{3}x + 36 = 36$$

$$4x^2 - 12\sqrt{3}x = 0$$

$$4x(x - 3\sqrt{3}) = 0,\ \text{donc } x = 0 \text{ ou } x = 3\sqrt{3}$$

En posant $x = 3\sqrt{3}$, nous trouvons
$y = -\sqrt{3}(3\sqrt{3}) + 6 = -3$,
ainsi les coordonnées du point T sont $T(3\sqrt{3}, -3)$.

$$A_1 = \text{Aire du quart de cercle} - \text{Aire du } \Delta\text{POQ}$$

$$= \frac{\pi(6)^2}{4} - \frac{6(2\sqrt{3})}{2}$$

donc $A_1 = 9\pi - 6\sqrt{3}$

$A_2 = B_1 + B_2$, où

$$B_1 = \frac{(3\sqrt{3} - 2\sqrt{3})3}{2} = \frac{3\sqrt{3}}{2} \text{ et}$$

$$B_2 = \int_{3\sqrt{3}}^{6}\sqrt{36 - x^2}\,dx$$

$$= \left(\frac{x\sqrt{36 - x^2}}{2} + 18\,\text{Arc sin}\left(\frac{x}{6}\right)\right)\Bigg|_{3\sqrt{3}}^{6}$$

$$= (0 + 9\pi) - \left(\frac{9\sqrt{3}}{2} + 6\pi\right)$$

$$= 3\pi - \frac{9\sqrt{3}}{2}$$

donc $A_2 = 3\pi - 3\sqrt{3}$

$A_3 = C_1 + C_2$, où

$$C_1 = \text{Aire du rectangle OENM} - \int_{0}^{3\sqrt{3}}\sqrt{36 - x^2}\,dx$$

$$= (3\sqrt{3})6 - \left(\frac{x\sqrt{36 - x^2}}{2} + 18\,\text{Arc sin}\left(\frac{x}{6}\right)\right)\Bigg|_{0}^{3\sqrt{3}}$$

$$= 18\sqrt{3} - \left(\frac{9\sqrt{3}}{2} + 6\pi\right)$$

$$= \frac{27\sqrt{3}}{2} - 6\pi,\ \text{et}$$

$$C_2 = \frac{3(4\sqrt{3} - 3\sqrt{3})}{2} = \frac{3\sqrt{3}}{2}$$

donc $A_3 = 15\sqrt{3} - 6\pi$

ainsi $A = A_1 + A_2 + A_3$

$$= (9\pi - 6\sqrt{3}) + (3\pi - 3\sqrt{3}) + (15\sqrt{3} - 6\pi)$$

$$= 6\pi + 6\sqrt{3}$$

d'où $A = (6\pi + 6\sqrt{3})\ \text{u}^2 \approx 29{,}24\ \text{u}^2$

c) En posant $y = 10 - \sqrt{100 - x^2}$ dans $(x-5)^2 + y^2 = 25$, nous obtenons

$$\begin{aligned}(x-5)^2 + \left(10 - \sqrt{100-x^2}\right)^2 &= 25\\ x^2 - 10x + 25 + 100 - 20\sqrt{100-x^2} + (100 - x^2) &= 25\\ -10x - 20\sqrt{100-x^2} + 200 &= 0\\ (20 - x) &= 2\sqrt{100-x^2}\\ 400 - 40x + x^2 &= 4(100 - x^2)\\ 5x^2 - 40x &= 0\\ 5x(x-8) &= 0\end{aligned}$$

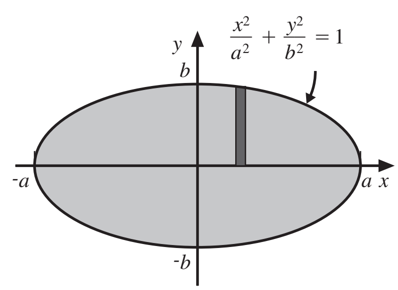
donc $x = 0$ ou $x = 8$

$$\begin{aligned}A &= \int_0^8 \left[\sqrt{25-(x-5)^2} - \left(10 - \sqrt{100-x^2}\right)\right]dx\\ &= \left[\frac{(x-5)\sqrt{25-(x-5)^2}}{2} + \frac{25}{2}\operatorname{Arc\,sin}\left(\frac{x-5}{5}\right) + \frac{x\sqrt{100-x^2}}{2} + 50\operatorname{Arc\,sin}\left(\frac{x}{10}\right) - 10x\right]\Bigg|_0^8\\ &= \left[\left(6 + \frac{25}{2}\operatorname{Arc\,sin}\left(\frac{3}{5}\right) + 24 + 50\operatorname{Arc\,sin}\left(\frac{4}{5}\right) - 80\right) - \left(\frac{25}{2}\operatorname{Arc\,sin}(-1) - 0\right)\right]\end{aligned}$$

$$A = \frac{25\pi}{4} + \frac{25}{2}\operatorname{Arc\,sin}\left(\frac{3}{5}\right) + 50\operatorname{Arc\,sin}\left(\frac{4}{5}\right) - 50$$

d'où $A \approx 24{,}04$ u²

7. a)



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}} = \pm\frac{b}{a}\sqrt{a^2 - x^2}$$

$$\begin{aligned}A &= 4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx\\ &= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx\\ &= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\operatorname{Arc\,sin}\left(\frac{x}{a}\right)\right]\Bigg|_0^a \quad \text{(table d'intégration)}\\ &= \frac{4b}{a}\left[\left(0 + \frac{a^2}{2}\operatorname{Arc\,sin} 1\right) - \left(0 + \frac{a^2}{2}\operatorname{Arc\,sin} 0\right)\right]\\ &= \frac{4b}{a}\left[\frac{a^2}{2}\frac{\pi}{2} - 0\right]\\ &= (\pi ab)\ \text{u}^2\end{aligned}$$

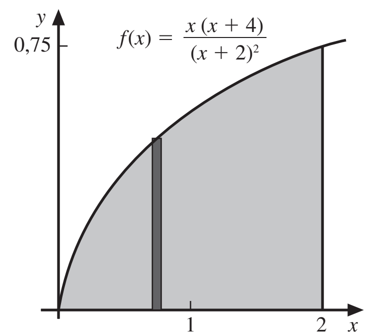
b) Grande ellipse : $a = 8$ et $b = 4$
Petite ellipse : $a = 5$ et $b = 3$

$$\begin{aligned}A &= \text{Aire entre les demi-ellipses}\\ &= \frac{1}{2}(\pi(8)(4) - \pi(5)(3))\\ &= 8{,}5\pi\ \text{u}^2\end{aligned}$$

$$\begin{aligned}\text{Volume} &= A(25)\\ &= (8{,}5\pi)(25)\\ &= 212{,}5\pi\end{aligned}$$

d'où $V \approx 667{,}59$ m³

8. a)



$$A = \int_0^2 \frac{x(x+4)}{(x+2)^2}\,dx$$

Puisque $\frac{x^2+4x}{x^2+4x+4} = 1 - \frac{4}{(x+2)^2}$, alors

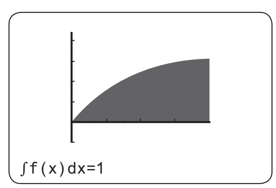
$$A = \int_0^2 \left(1 - \frac{4}{(x+2)^2}\right)dx = \left(x + \frac{4}{(x+2)}\right)\Bigg|_0^2 = 1\ \text{u}^2$$

Vérification du résultat

```
Plot1 Plot2 Plot3
\Y1=(X(X+4))/
(X+2)^2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=2
Xscl=.5
Ymin=0
Ymax=1
Yscl=.25
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



b) $\int_0^2 f(x)\,dx = f(c)\cdot(2-0)$, où $c \in [0, 2]$

$$1 = \frac{c(c+4)}{(c+2)^2}(2)$$

$$c^2 + 4c + 4 = 2c^2 + 8c$$
$$c^2 + 4c - 4 = 0$$

d'où $c = \left(2\sqrt{2} - 2\right)$ $\left(\left(-2 - 2\sqrt{2}\right)\text{ à rejeter}\right)$

9. a) $A = \int_0^{\frac{\pi}{3}} 2\sin 3x\,dx = \left.\frac{-2\cos 3x}{3}\right|_0^{\frac{\pi}{3}} = \frac{4}{3}$

$\bar{x} = \frac{1}{A}\int_0^{\frac{\pi}{3}} xy\,dx = \frac{2}{A}\int_0^{\frac{\pi}{3}} x\sin 3x\,dx$

Calculons $I = \int x\sin 3x\,dx$.

$u = x$	$dv = \sin 3x\,dx$
$du = dx$	$v = \frac{-\cos 3x}{3}$

$I = \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\,dx$

$= \frac{-x\cos 3x}{3} + \frac{\sin 3x}{9} + C$

$\bar{x} = \frac{2}{A}\left.\left(\frac{-x\cos 3x}{3} + \frac{\sin 3x}{9}\right)\right|_0^{\frac{\pi}{3}} = \frac{6}{4}\left(\frac{\pi}{9}\right) = \frac{\pi}{6}$

$\bar{y} = \frac{1}{2A}\int_0^{\frac{\pi}{3}} y^2\,dx$

$= \frac{4}{2A}\int_0^{\frac{\pi}{3}} \sin^2 3x\,dx$

$= \frac{2}{A}\int_0^{\frac{\pi}{3}}\left(\frac{1-\cos 6x}{2}\right)dx$

$= \frac{2}{A}\left.\left(x - \frac{\sin 6x}{6}\right)\right|_0^{\frac{\pi}{3}}$

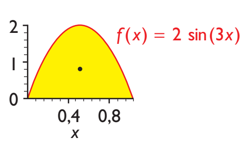
$= \frac{6}{4}\left(\frac{\pi}{6}\right) = \frac{\pi}{4}$

d'où $C(\bar{x}, \bar{y}) = C\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$

```
> f:=x->2*sin(3*x);
```

$f := x \to 2\sin(3x)$

```
> with(plots):
> c1:=plot(f(x),x=0..Pi/3,color=orange):
> c2:=plot(f(x),x=0..Pi/3,filled=true,color=yellow):
> c:=plot([[Pi/6,Pi/4]],style=point,symbol=circle,
  color=black):
> display(c1,c2,c);
```



b) $A = \int_0^1 e^x\,dx = e^x\,\big|_0^1 = (e-1)$

$\bar{x} = \frac{1}{A}\int_0^1 xy\,dx = \frac{1}{A}\int_0^1 xe^x\,dx = \left.\frac{1}{A}(xe^x - e^x)\right|_0^1 = \frac{1}{e-1}$

$\bar{y} = \frac{1}{2A}\int_0^1 y^2\,dx$

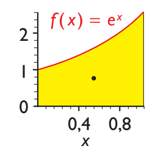
$= \frac{1}{2A}\int_0^1 e^{2x}\,dx$

$= \left.\frac{1}{2A}\,\frac{e^{2x}}{2}\right|_0^1$

$= \frac{1}{2(e-1)}\left(\frac{e^2-1}{2}\right) = \frac{e+1}{4}$

d'où $C(\bar{x}, \bar{y}) = C\left(\frac{1}{e-1}, \frac{e+1}{4}\right)$

```
> f:=x->exp(x):
> with(plots):
> c1:=plot(f(x),x=0..1,color=orange):
> c2:=plot(f(x),x=0..1,filled=true,color=yellow):
> c:=plot([[1/(exp(1)-1),(exp(1)+1)/4]],style=point,
  symbol=circle,color=black):
> display(c1,c2,c);
```



c) Soit la fonction $x = \ln y$ où $y \in [1, e]$.

$A = \int_1^e \ln y\,dy$

Calculons $I = \int \ln y\,dy$.

$u = \ln y$	$dv = dy$
$du = \frac{1}{y}\,dy$	$v = y$

$I = y\ln y - \int 1\,dy = y\ln y - y + C$

$A = \int_1^e \ln y\,dy = (y\ln y - y)\,\big|_1^e = 1$

$\bar{x} = \frac{1}{2A}\int_1^e x^2\,dy = \frac{1}{2A}\int_1^e (\ln y)^2\,dy$

Calculons $I = \int (\ln y)^2\,dy$.

$u = (\ln y)^2$	$dv = dy$
$du = \frac{2(\ln y)}{y}\,dy$	$v = y$

$I = y(\ln y)^2 - 2\int \ln y\,dy$

$= y(\ln y)^2 - 2[y\ln y - y] + C$

$\bar{x} = \frac{1}{2A}\int_1^e (\ln y)^2\,dy$

$= \left.\frac{1}{2A}\left(y(\ln y)^2 - 2y\ln y + 2y\right)\right|_1^e$

$= \frac{e-2}{2}$

$\bar{y} = \frac{1}{A}\int_1^e yx\,dy = \frac{1}{A}\int_1^e y\ln y\,dy$

Calculons $I = \int y\ln y\,dy$.

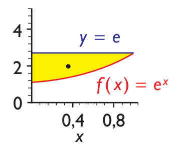
$u = \ln y$	$dv = y\,dy$
$du = \frac{1}{y}\,dy$	$v = \frac{y^2}{2}$

$$I = \frac{y^2}{2}\ln y - \frac{1}{2}\int y\,dy = \frac{y^2}{2}\ln y - \frac{y^2}{4} + C$$

$$\overline{y} = \frac{1}{A}\int_1^e y\ln y\,dy = \frac{1}{A}\left(\frac{y^2}{2}\ln y - \frac{y^2}{4}\right)\Bigg|_1^e = \frac{e^2+1}{4}$$

d'où $C(\overline{x}, \overline{y}) = C\left(\frac{e-2}{2}, \frac{e^2+1}{4}\right)$

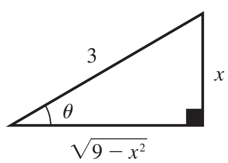
```
> f:=x->exp(x):
> g:=x->exp(1):
> with(plots):
> c1:=plot([f(x),g(x)],x=0..1,color=[orange,blue]):
> c2:=plot([max(min(f(x),g(x)),0),min(max(f(x),g(x)),0),
  f(x),g(x)],x=0..1,y=0..4,filled=true,color=[white,white,
  yellow,yellow]):
> c:=plot([[(exp(1)-2)/2,((exp(2))+1)/4]],style=point,
  symbol=circle,color=black):
> display(c1,c2,c);
```



d) $A = \int_0^2 \frac{x}{9-x^2}\,dx = \frac{-1}{2}\ln|9-x^2|\Big|_0^2 = \frac{2\ln 3 - \ln 5}{2}$

$$\overline{x} = \frac{1}{A}\int_0^2 xy\,dx = \frac{1}{A}\int_0^2 \frac{x^2}{9-x^2}\,dx$$

$$x = 3\sin\theta \qquad dx = 3\cos\theta\,d\theta \qquad \theta = \text{Arc}\sin\left(\frac{x}{3}\right)$$

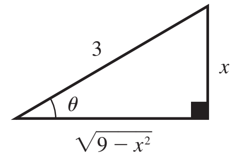


$$\begin{aligned}
\text{Calculons } I &= \int \frac{x^2}{9-x^2}\,dx \\
&= \int \frac{9\sin^2\theta}{9-9\sin^2\theta}3\cos\theta\,d\theta \\
&= 3\int \frac{\sin^2\theta}{\cos\theta}\,d\theta \\
&= 3\int \frac{1-\cos^2\theta}{\cos\theta}\,d\theta \\
&= 3\left[\int \sec\theta\,d\theta - \int\cos\theta\,d\theta\right] \\
&= 3\left[\ln|\sec\theta + \tan\theta| - \sin\theta\right] + C \\
&= 3\left[\ln\left|\frac{x+3}{\sqrt{9-x^2}}\right| - \frac{x}{3}\right] + C
\end{aligned}$$

$$\overline{x} = \frac{3}{A}\left[\ln\left|\frac{x+3}{\sqrt{9-x^2}}\right| - \frac{x}{3}\right]\Bigg|_0^2 = \frac{3\ln 5 - 4}{2\ln 3 - \ln 5}$$

$$\overline{y} = \frac{1}{2A}\int_0^2 y^2\,dx = \frac{1}{2A}\int_0^2 \frac{x^2}{(9-x^2)^2}\,dx$$

$$x = 3\sin\theta \qquad dx = 3\cos\theta\,d\theta \qquad \theta = \text{Arc}\sin\left(\frac{x}{3}\right)$$



$$\begin{aligned}
\text{Calculons } I &= \int \frac{x^2}{(9-x^2)^2}\,dx \\
&= \int \frac{9\sin^2\theta}{(9-9\sin^2\theta)^2}3\cos\theta\,d\theta \\
&= \frac{1}{3}\int \frac{\sin^2\theta}{\cos^3\theta}\,d\theta \\
&= \frac{1}{3}\int \frac{1-\cos^2\theta}{\cos^3\theta}\,d\theta \\
&= \frac{1}{3}\left[\int \sec^3\theta\,d\theta - \int\sec\theta\,d\theta\right] \\
&= \frac{1}{3}\left[\frac{\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|}{2} - \ln|\sec\theta + \tan\theta|\right] + C \\
&= \frac{1}{6}\left[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\right] + C \\
&= \frac{1}{6}\left[\frac{3x}{9-x^2} - \ln\left|\frac{x+3}{\sqrt{9-x^2}}\right|\right] + C
\end{aligned}$$

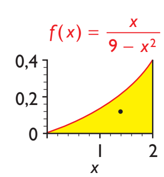
$$\overline{y} = \frac{1}{12A}\left[\frac{3x}{9-x^2} - \ln\left|\frac{x+3}{\sqrt{9-x^2}}\right|\right]\Bigg|_0^2 = \frac{12 - 5\ln 5}{60(2\ln 3 - \ln 5)}$$

d'où $C(\overline{x}, \overline{y}) = C\left(\frac{3\ln 5 - 4}{2\ln 3 - \ln 5}, \frac{12 - 5\ln 5}{60(2\ln 3 - \ln 5)}\right)$

```
> f:=x->x/(9-x^2);
```

$$f := x \to \frac{x}{9-x^2}$$

```
> with(plots):
> c1:=plot(f(x),x=0..2,color=orange):
> c2:=plot(f(x),x=0..2,filled=true,color=yellow):
> c:=plot([[(3*ln(5)-4)/(2*ln(3)-ln(5)),(12-5*ln(5))/
  (60*(2*ln(3)-ln(5)))]],style=point,symbol=circle,
  color=black):
> display(c1,c2,c);
```



e) $A = \int_0^{2\pi}(1+\cos x)\,dx = (x - \sin x)\Big|_0^{2\pi} = 2\pi$

$$\begin{aligned}
\overline{x} &= \frac{1}{A}\int_0^{2\pi} xy\,dx \\
&= \frac{1}{A}\int_0^{2\pi} x(1+\cos x)\,dx \\
&= \frac{1}{A}\int_0^{2\pi}(x + x\cos x)\,dx \\
&= \frac{1}{A}\left(\frac{x^2}{2} + x\sin x + \cos x\right)\Bigg|_0^{2\pi} = \frac{1}{2\pi}(2\pi^2) = \pi
\end{aligned}$$

4

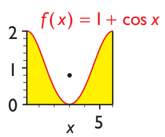
$$\begin{aligned}\overline{y} &= \frac{1}{2A}\int_0^{2\pi} y^2\,dx\\ &= \frac{1}{2A}\int_0^{2\pi}(1+\cos x)^2\,dx\\ &= \frac{1}{2A}\int_0^{2\pi}(1+2\cos x+\cos^2 x)\,dx\\ &= \frac{1}{2A}\int_0^{2\pi}\left(1+2\cos x+\frac{1+\cos 2x}{2}\right)dx\\ &= \frac{1}{2A}\left(x+2\sin x+\frac{x}{2}+\frac{\sin 2x}{4}\right)\Bigg|_0^{2\pi}\\ &= \frac{1}{4\pi}(3\pi) = \frac{3}{4}\end{aligned}$$

d'où $C(\overline{x}, \overline{y}) = C\left(\pi, \frac{3}{4}\right)$

```
> f:=x->1+cos(x);
```

$$f := x \rightarrow 1 + \cos(x)$$

```
> with(plots):
> c1:=plot(f(x),x=0..2*Pi,color=orange):
> c2:=plot(f(x),x=0..2*Pi,filled=true,color=yellow):
> c:=plot([[Pi,3/4]],style=point,symbol=circle,
  color=black):
> display(c1,c2,c);
```



f)
```
> A:=int(f(x)-g(x),x=-2..2);
```

$$A := 2\pi$$

```
> x1:=(1/A)*(int(x*(f(x)-g(x)),x=-2..2));
```

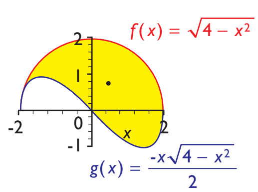
$$x1 := \frac{1}{2}$$

```
> y1:=(1/(2*A))*(int((f(x)^2-g(x)^2),x=-2..2));
```

$$y1 := \frac{32}{15}\frac{1}{\pi}$$

d'où $C(\overline{x}, \overline{y}) = C\left(\frac{1}{2}, \frac{32}{15\pi}\right)$

```
> with(plots):
> c1:=plot([f(x),g(x)],x=-2..2,color=[orange,blue]):
> c2:=plot([max(min(f(x),g(x)),0),min(max(f(x),g(x)),0),
  f(x),g(x)],x=-2..2,y=-1..2,filled=true,color=[white,white,
  yellow,yellow],scaling=constrained):
> c:=plot([[1/2,32/(15*Pi)]],style=point,symbol=circle,
  color=black):
> display(c1,c2,c);
```



10. a) $\frac{dv}{dt} = a$

$\int dv = \int t\cos t\,dt$

$u = t$ $du = dt$	$dv = \cos t\,dt$ $v = \sin t$

$v = t\sin t - \int \sin t\,dt$

$v = t\sin t + \cos t + C$

En remplaçant t par 0 et v par 0, nous obtenons

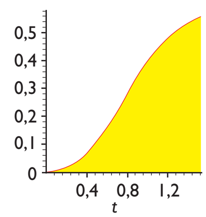
$0 = 0 + 1 + C$, donc $C = -1$

d'où $v(t) = t\sin t + \cos t - 1$

b)
```
> v:=t->t*sin(t)+cos(t)-1;
```

$$v := t \rightarrow t\sin(t) + \cos(t) - 1$$

```
> with(plots):
> c1:=plot(v(t),t=0..Pi/2,color=orange):
> c11:=plot(v(t),t=0..Pi/2,filled=true,color=yellow):
> display(c1,c11);
```



$d_1 = \int_0^{\frac{\pi}{2}} v(t)\,dt$

$\int v(t)\,dt = \int t\sin t\,dt + \int \cos t\,dt - \int dt$

$u = t$ $du = dt$	$dv = \sin t\,dt$ $v = -\cos t\,dt$

$= -t\cos t + \int \cos t\,dt + \sin t - t$

donc $\int v(t)\,dt = -t\cos t + 2\sin t - t + C$

$$\begin{aligned} d_1 &= \int_0^{\frac{\pi}{2}} v(t)\,dt\\ &= (-t\cos t + 2\sin t - t)\Big|_0^{\frac{\pi}{2}}\\ &= \left(2 - \frac{\pi}{2}\right)\end{aligned}$$

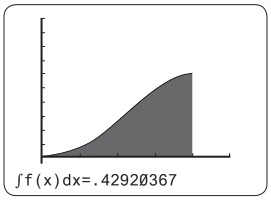
d'où $d_1 = \left(2 - \frac{\pi}{2}\right)$ m $\approx 0{,}43$ m

Vérification du résultat

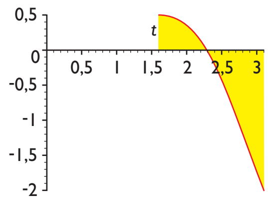
```
Plot1 Plot2 Plot3
\Y1■Xsin(X)+cos
(X)-1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=1.5707964…
Xscl=.4
Ymin=0
Ymax=1
Yscl=.1
Xres=1
```

```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```



c) > c2:=plot(v(t),t=Pi/2..Pi,view=[0..Pi,v(Pi/2)..v(Pi)], color=orange):
> c22:=plot(v(t),t=Pi/2..Pi,filled=true,color=yellow):
> display(c2,c22);



> t1:=fsolve(v(t)=0,t=Pi/2..Pi);

$$t1 := 2.331122370$$

> dist2:=Int(v(t),t=Pi/2..t1)+Int(-v(t),t=t1..Pi);

$$dist2 := \int_{\frac{\pi}{2}}^{2.331122370} t\sin(t) + \cos(t) - 1\,dt + \int_{2.331122370}^{\pi} -t\sin(t) - \cos(t) + 1\,dt$$

> A1:=int(v(t),t=Pi/2..t1);

$$A1 := 0.2954076806$$

> A2:=int(-v(t),t=t1..Pi);

$$A2 := 0.7246113538$$

> dist2:=A1+A2;

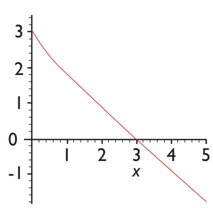
$$dist2 := 1.020019034$$

d'où $d_2 \approx 1{,}02$ m

11. a) $v(t) = \dfrac{(3-t)(t+2)^2}{(t+1)(t+4)}$

$$= -t + 4 + \frac{4}{3(t+1)} - \frac{28}{3(t+4)}$$

> plot((3-t)*((t+2)^2)/((t+1)*(t+4)),t=0..5,color=orange);



$$d_{[0\,s,\,3\,s]} = \int_0^3 \left(-t + 4 + \frac{4}{3(t+1)} - \frac{28}{3(t+4)}\right) dt$$

$$= \left(\frac{-t^2}{2} + 4t + \frac{4\ln|t+1|}{3} - \frac{28\ln|t+4|}{3}\right)\Bigg|_0^3$$

$$= \frac{15}{2} + \frac{32\ln 4}{3} - \frac{28\ln 7}{3}$$

d'où $d_{[0\,s,\,3\,s]} \approx 4{,}13$ m

$$d_{[3\,s,\,5\,s]} = \left[-\left(\frac{-t^2}{2} + 4t + \frac{4\ln|t+1|}{3} - \frac{28\ln|t+4|}{3}\right)\right]\Bigg|_3^5$$

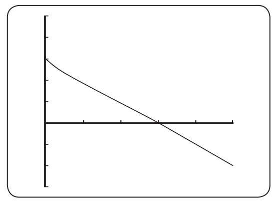
$$\approx 1{,}80$$

d'où $d_{[3\,s,\,5\,s]} \approx 1{,}80$ m

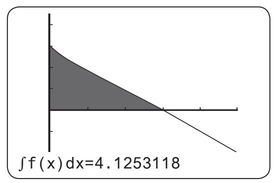
Vérification du résultat

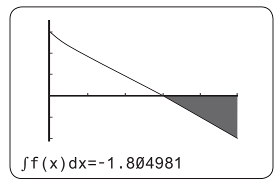
```
Plot1 Plot2 Plot3
\Y1■((3-X)(X+2)²/
((X+1)(X+4)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=5
Xscl=1
Ymin=-3
Ymax=3
Yscl=1
Xres=1
```



```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```





b) $a(t) = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(-t + 4 + \dfrac{4}{3(t+1)} - \dfrac{28}{3(t+4)}\right)$

$$= \left(-1 - \frac{4}{3(t+1)^2} + \frac{28}{3(t+4)^2}\right)$$

$$a(2) = \left(-1 - \frac{4}{27} + \frac{28}{108}\right) = \frac{-8}{9}$$

d'où $a(2) = \dfrac{-8}{9}$ m/s² $= -0{,}\bar{8}$ m/s²

12. $R = \int R_m(q)\,dq$

$$= \int 10^3(2q - qe^{-0{,}5q})\,dq$$

$$= 10^3\left[q^2 - \int qe^{-0{,}5q}\,dq\right]$$

Déterminons $\int qe^{-0{,}5q}\,dq$.

$u = q$	$dv = e^{-0{,}5q}\,dq$
$du = dq$	$v = -2e^{-0{,}5q}$

Ainsi $\int qe^{-0{,}5q}\,dq = -2qe^{-0{,}5q} + 2\int e^{-0{,}5q}\,dq$

$$= -2qe^{-0{,}5q} - 4e^{-0{,}5q}$$

donc $R(q) = 10^3[q^2 + 2qe^{-0{,}5q} + 4e^{-0{,}5q}] + C$

En supposant que $R(q) = 0$ lorsque $q = 0$, on trouve $C = -4000$, ainsi le revenu cherché est donné par $R(6) - R(0) \approx 32\ 796{,}59$ \$.

4

13. a) $F(T) = e^{iT}\int_0^T a(t)\,e^{-it}\,dt$

En posant $T = 5$, $a(t) = 4000$ et $i = 0{,}045$, nous obtenons

$$F(5) = e^{(0{,}045)5}\int_0^5 4000e^{-0{,}045t}\,dt$$

$$= \frac{-4000e^{0{,}225}}{0{,}045}\,e^{-0{,}045t}\Big|_0^5$$

$$= \frac{-4000e^{0{,}225}}{0{,}045}\,(e^{-0{,}225} - 1)$$

$$= 22\,428{,}686\ldots$$

d'où environ 22 428,69 $

b) En posant $T = 10$, $a(t) = (2500 + 200t)$ et $i = 0{,}04$, nous obtenons

$$F(10) = e^{(0{,}04)10}\int_0^{10}(2500 + 200t)e^{-0{,}04t}\,dt$$

Calculons d'abord $I = \int(2500 + 200t)e^{-0{,}04t}\,dt$.

$u = 2500 + 200t$ $\qquad dv = e^{-0{,}04t}\,dt$

$du = 200\,dt$ $\qquad v = \dfrac{-e^{-0{,}04t}}{0{,}04} = -25e^{-0{,}04t}$

$$I = (2500 + 200t)(-25e^{-0{,}04t}) + 25(200)\int e^{-0{,}04t}\,dt$$

$$= (2500 + 200t)(-25e^{-0{,}04t}) - \frac{500e^{-0{,}04t}}{0{,}04} + C$$

$$= -62\,500e^{-0{,}04t} - 5000te^{-0{,}04t} - 125\,000e^{-0{,}04t} + C$$

$$= (-187\,500 - 5000t)e^{-0{,}04t} + C$$

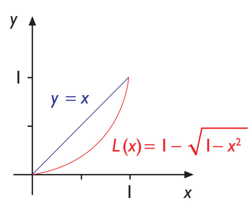
$$F(10) = e^{0{,}4}(-187\,500 - 5000t)e^{-0{,}04t}\Big|_0^{10}$$

$$= e^{0{,}4}(-237\,500e^{-0{,}4} + 187\,500)$$

$$\approx 42\,217{,}13$$

d'où environ 42 217,13 $

14. a) $L(x) = 1 - \sqrt{1 - x^2}$



b) $G = 2\int_0^1 \big(x - L(x)\big)\,dx$

$$= 2\int_0^1\Big[x - \big(1 - \sqrt{1 - x^2}\big)\Big]dx$$

$$= 2\int_0^1\big(x - 1 + \sqrt{1 - x^2}\big)\,dx$$

$$= 2\left(\frac{x^2}{2} - x + \frac{x\sqrt{1 - x^2}}{2} + \frac{1}{2}\,\text{Arc}\sin x\right)\Bigg|_0^1$$

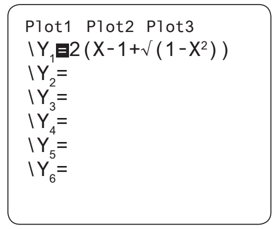
(table d'intégration)

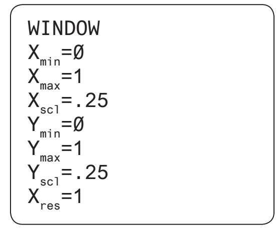
$$= 2\left[\left(\frac{1}{2} - 1 + \frac{\pi}{4}\right) - 0\right]$$

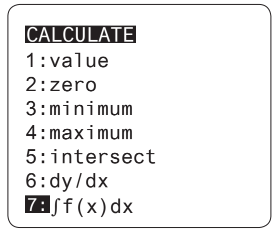
$$= \left(\frac{\pi}{2} - 1\right)$$

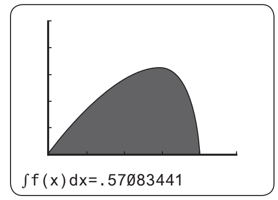
d'où $G \approx 0{,}57$

Vérification du résultat









15. a)

$$\frac{dy}{dt} = ky(N - y)$$

$$\frac{dy}{y(N - y)} = k\,dt$$

Décomposons $\dfrac{1}{y(N - y)}$ en une somme de fractions partielles.

$$\frac{1}{y(N - y)} = \frac{A}{y} + \frac{B}{N - y}$$

$$= \frac{A(N - y) + By}{y(N - y)}$$

$$1 = (B - A)y + AN$$

Ainsi $B - A = 0$

$AN = 1$

donc $A = \dfrac{1}{N}$ et $B = \dfrac{1}{N}$

$$\int\frac{1}{y(N - y)}\,dy = \int k\,dt$$

$$\int\left(\frac{1}{Ny} + \frac{1}{N(N - y)}\right)dy = kt + C_1$$

$$\frac{1}{N}\ln|y| - \frac{1}{N}\ln|N - y| = kt + C_1$$

$$\ln\left|\frac{y}{N - y}\right| = Nkt + C_2$$

$$\frac{y}{N - y} = e^{Nkt + C_2} = C_3e^{Nkt}$$

$$y = NC_3e^{Nkt} - yC_3e^{Nkt}$$

$$y(1 + C_3e^{Nkt}) = NC_3e^{Nkt}$$

$$y = \frac{NC_3e^{Nkt}}{1 + C_3e^{Nkt}}$$

$$= \frac{Ne^{Nkt}}{\dfrac{1}{C_3} + e^{Nkt}}$$

$$= \frac{N}{Ce^{-Nkt} + 1} \qquad \left(C = \frac{1}{C_3}\right)$$

d'où $y = \dfrac{N}{1 + Ce^{-Nkt}}$ (équation 1)

b) Déterminons le point d'inflexion.

$$\frac{dy}{dt} = \frac{0 - NC(-Nk)e^{-Nkt}}{(1 + Ce^{-Nkt})^2} = \frac{CN^2ke^{-Nkt}}{(1 + Ce^{-Nkt})^2}$$

$$\frac{d^2y}{dt^2} = \frac{CN^3k^2e^{-Nkt}(1 + e^{-Nkt})\left[-(1 + Ce^{-Nkt}) + 2Ce^{-Nkt}\right]}{(1 + Ce^{-Nkt})^4}$$

Au point d'inflexion $\frac{d^2y}{dt^2} = 0$

donc
$$\begin{aligned} -1 - Ce^{-Nkt} + 2Ce^{-Nkt} &= 0 \\ -1 + Ce^{-Nkt} &= 0 \\ Ce^{-Nkt} &= 1 \\ e^{Nkt} &= C \\ t &= \frac{\ln C}{Nk} \end{aligned}$$

En remplaçant t par $\frac{\ln C}{Nk}$ dans l'équation 1,

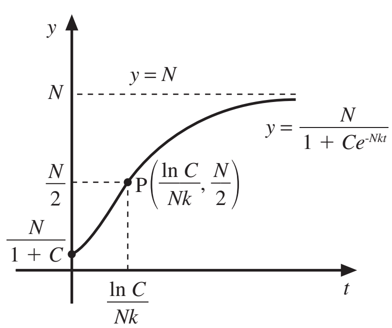
nous trouvons $\frac{N}{2}$,

d'où le point d'inflexion est le point $P\left(\frac{\ln C}{Nk}, \frac{N}{2}\right)$

Déterminons l'équation de l'asymptote horizontale.

$$\lim_{t \to +\infty} \frac{N}{1 + Ce^{-Nkt}} = \frac{N}{1 + 0} = N$$

d'où la droite d'équation $y = N$ est l'asymptote horizontale.



c) En remplaçant N par 1500 dans l'équation 1,

$$y = \frac{1500}{1 + Ce^{-1500kt}}$$

En remplaçant t par 0 et y par 300, nous obtenons

$$300 = \frac{1500}{1 + C}, \text{ donc } C = 4$$

ainsi $y = \frac{1500}{1 + 4e^{-1500kt}}$

En remplaçant t par 1 et y par 500, nous obtenons

$$\begin{aligned} 500 &= \frac{1500}{1 + 4e^{-1500k}} \\ 1 + 4e^{-1500k} &= 3 \\ e^{-1500k} &= \frac{1}{2} \\ -1500k &= \ln\left(\frac{1}{2}\right), \text{ donc } k = \frac{\ln 2}{1500} \end{aligned}$$

ainsi $y = \frac{1500}{1 + 4e^{-(\ln 2)t}}$ ou $y = \frac{1500}{1 + 4\left(\frac{1}{2}\right)^t}$

i) En remplaçant t par 2,5, $y \approx 878$ bactéries.

ii) En remplaçant y par (90 %(1500)),

$$90\,\%(1500) = \frac{1500}{1 + 4\left(\frac{1}{2}\right)^t}$$

$$1 + 4\left(\frac{1}{2}\right)^t = \frac{10}{9}, \text{ donc } \left(\frac{1}{2}\right)^t = \frac{1}{36}$$

$$t = \frac{\ln\left(\frac{1}{36}\right)}{\ln\left(\frac{1}{2}\right)} = 5{,}169\ldots$$

d'où $t \approx 5{,}17$ heures

iii) En remplaçant C par 4, N par 1500

et k par $\frac{\ln 2}{1500}$, dans $P\left(\frac{\ln C}{Nk}, \frac{N}{2}\right)$

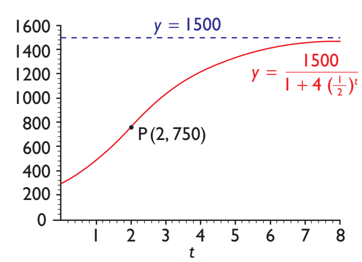
nous obtenons le point d'inflexion P(2, 750)

La droite d'équation $y = 1500$ est une asymptote horizontale.

```
> N:=t->1500/(1+4*(1/2)^t);
```

$$N := t \to \frac{1500}{1 + 4\left(\frac{1}{2}\right)^t}$$

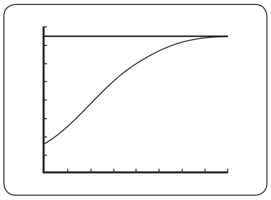
```
> c1:=plot(N(t),t=0..8,color=orange):
> ah:=plot(1500,t=0..8,color=blue,linestyle=4):
> P:=plot([[ln(4)/ln(2),750]],symbol=circle,
  style=point,color=orange):
> display(c1,ah,P,view=[0..8,0..1600]);
```



Vérification du résultat

```
Plot1 Plot2 Plot3
\Y1=1500
\Y2=1500/(1+4(.5)^X)
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=8
Xscl=1
Ymin=0
Ymax=1600
Yscl=200
Xres=1
```



16. a) Soit h la hauteur de la plante en cm et t, le temps en jours.

$$\frac{dh}{dt} = kh(60-h)$$

$$\frac{dh}{h(60-h)} = k\,dt$$

$$\int \frac{1}{h(60-h)}\,dh = \int k\,dt$$

Décomposons $\dfrac{1}{h(60-h)}$ en une somme de fractions partielles.

$$\frac{1}{h(60-h)} = \frac{A}{h} + \frac{B}{(60-h)} = \frac{\frac{1}{60}}{h} + \frac{\frac{1}{60}}{(60-h)}$$

Donc $\displaystyle\int\left(\frac{1}{60h} + \frac{1}{60(60-h)}\right)dh = \int k\,dt$

$$\frac{1}{60}\ln h - \frac{1}{60}\ln(60-h) = kt + C$$

$$\frac{1}{60}\ln\left(\frac{h}{60-h}\right) = kt + C$$

Puisque à $t = 0$, $h = 10$, alors

$$\frac{1}{60}\ln\left(\frac{10}{50}\right) = k(0) + C,\ \text{donc } C = \frac{1}{60}\ln\left(\frac{1}{5}\right)$$

$$\frac{1}{60}\ln\left(\frac{h}{60-h}\right) = kt + \frac{1}{60}\ln\left(\frac{1}{5}\right)$$

Puisque à $t = 3$, $h = 12$, alors

$$\frac{1}{60}\ln\left(\frac{12}{48}\right) = 3k + \frac{1}{60}\ln\left(\frac{1}{5}\right),\ \text{donc } k = \frac{1}{180}\ln\left(\frac{5}{4}\right)$$

ainsi $\displaystyle\frac{1}{60}\ln\left(\frac{h}{60-h}\right) = \frac{1}{180}\ln\left(\frac{5}{4}\right)t + \frac{1}{60}\ln\left(\frac{1}{5}\right)$

$$\ln\left(\frac{h}{60-h}\right) = \frac{1}{3}\ln\left(\frac{5}{4}\right)t + \ln\left(\frac{1}{5}\right)$$

$$\frac{h}{60-h} = \frac{1}{5}e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}$$

$$h = 12e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t} - \frac{h}{5}e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}$$

$$h + \frac{h}{5}e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t} = 12e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}$$

$$h\left[1 + \frac{e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}}{5}\right] = 12e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}$$

$$h = \frac{12e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}}{1 + \frac{e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}}{5}}$$

$$= \frac{60e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}}{5 + e^{\frac{1}{3}\ln\left(\frac{5}{4}\right)t}}$$

$$= \frac{60}{1 + 5e^{\frac{-1}{3}\ln\left(\frac{5}{4}\right)t}}$$

d'où $\displaystyle h(t) = \frac{60}{1 + 5\left(\frac{4}{5}\right)^{\frac{t}{3}}}$

b) $\displaystyle h(14) = \frac{60}{1 + 5\left(\frac{4}{5}\right)^{\frac{14}{3}}} \approx 21{,}7$ cm

c) Pour déterminer la hauteur maximale de la plante,

il s'agit de calculer $\displaystyle\lim_{t\to+\infty}\left(\frac{60}{1 + 5\left(\frac{4}{5}\right)^{\frac{t}{3}}}\right)$.

Puisque $\displaystyle\lim_{t\to+\infty}\left(\frac{60}{1 + 5\left(\frac{4}{5}\right)^{\frac{t}{3}}}\right) = 60$, alors il faut calculer le nombre de jours pour que la plante atteigne 30 cm.

Ainsi $\displaystyle 30 = \frac{60}{1 + 5\left(\frac{4}{5}\right)^{\frac{t}{3}}}$

$$1 + 5\left(\frac{4}{5}\right)^{\frac{t}{3}} = 2$$

$$\left(\frac{4}{5}\right)^{\frac{t}{3}} = \frac{1}{5}$$

$$t = \frac{3\ln\left(\frac{1}{5}\right)}{\ln\left(\frac{4}{5}\right)}$$

$$t \approx 21{,}6 \text{ jours}$$

d) `> h:=t->60/(1+5*(4/5)^(t/3));`

$$h := t \to 60\,\frac{1}{1 + 5\left(\frac{4}{5}\right)^{\frac{1}{3}t}}$$

> diff(h(t),t);

$$-100\,\frac{\left(\frac{4}{5}\right)^{\frac{1}{3}t}\ln\left(\frac{4}{5}\right)}{\left(1+5\left(\frac{4}{5}\right)^{\frac{1}{3}t}\right)^2}$$

> h1:=t->diff(h(t),t);

$$h1 := t \rightarrow \text{diff}\left(h(t),\, t\right)$$

> diff(h1(t),t);

$$\frac{1000}{3}\,\frac{\left(\left(\frac{4}{5}\right)^{\frac{1}{3}t}\right)^2\ln\left(\frac{4}{5}\right)^2}{\left(1+5\left(\frac{4}{5}\right)^{\frac{1}{3}t}\right)^3}-\frac{100}{3}\,\frac{\left(\frac{4}{5}\right)^{\frac{1}{3}t}\ln\left(\frac{4}{5}\right)^2}{\left(1+5\left(\frac{4}{5}\right)^{\frac{1}{3}t}\right)^2}$$

> h2:=t->diff(h1(t),t);

$$h2 := t \rightarrow \text{diff}\left(h1(t),\, t\right)$$

> a:=solve(h2(t)=0);

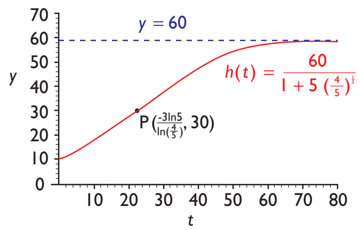
$$a := -3\,\frac{\ln(5)}{\ln\left(\frac{4}{5}\right)}$$

> b:=evalf(h(-3*ln(5)/ln(4/5)));

$$b := 30.00000000$$

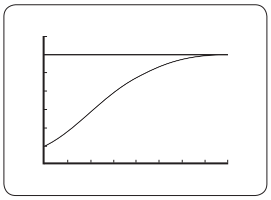
Le point d'inflexion est le point $P\left(\dfrac{-3\ln 5}{\ln\left(\frac{4}{5}\right)},\, 30\right)$

> with(plots):
> c1:=plot(h(t),t=0..80,y=0..70,color=orange):
> ah:=plot(60,t=0..80,color=blue,linestyle=4):
> P:=plot([[-3*ln(5)/ln(4/5),30]],symbol=circle,
> style=point,color=orange):
> display(c1,ah,P);




```
Plot1 Plot2 Plot3
\Y1■60
\Y2■60/(1+5(.8)^
(X/3)
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
Xmin=0
Xmax=80
Xscl=10
Ymin=0
Ymax=70
Yscl=10
Xres=1
```

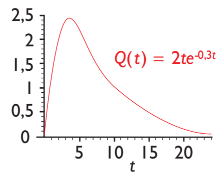


e) Le taux de croissance est le plus rapide lorsque $h''(t) = 0$, c'est-à-dire au point d'inflexion, d'où la hauteur de la plante est de 30 cm.

17. a) > Q:=t->2*t*exp(-0.3*t);

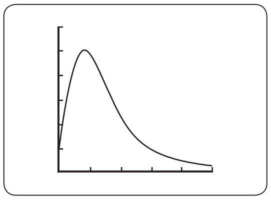
$$Q := t \rightarrow 2te^{(-0.3t)}$$

> plot(Q(t),t=0..24,color=orange);




```
Plot1 Plot2 Plot3
\Y1■2Xe^(-.3X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
Xmin=0
Xmax=25
Xscl=5
Ymin=0
Ymax=3
Yscl=.5
Xres=1
```



b) Calculons d'abord $I = \int 2te^{-0,3t}\,dt$.

$u = 2t$	$dv = e^{-0,3t}\,dt$
$du = 2\,dt$	$v = \dfrac{-e^{-0,3t}}{0,3}$

$$I = \frac{-2te^{-0,3t}}{0,3} + \frac{2}{0,3}\int e^{-0,3t}\,dt$$

$$I = \frac{-2te^{-0,3t}}{0,3} - \frac{2e^{-0,3t}}{0,09} + C$$

i) $$Q_{[0\,\text{h},\,6\,\text{h}]} = \frac{1}{6}\int_0^6 2te^{-0,3t}\,dt = \frac{1}{6}\left[\frac{-2te^{-0,3t}}{0,3} - \frac{2e^{-0,3t}}{0,09}\right]\Bigg|_0^6$$

d'où environ 1,989 mg/cm³

ii) $$Q_{[18\,\text{h},\,24\,\text{h}]} = \frac{1}{6}\left[\frac{-2te^{-0,3t}}{0,3} - \frac{2e^{-0,3t}}{0,09}\right]\Bigg|_{18}^{24}$$

d'où environ 0,084 mg/cm³

iii) $$Q_{[0\,\text{h},\,24\,\text{h}]} = \frac{1}{24}\left[\frac{-2te^{-0,3t}}{0,3} - \frac{2e^{-0,3t}}{0,09}\right]\Bigg|_0^{24}$$

d'où environ 0,920 mg/cm³

18. a) $f(x) = k(x + 2)e^{-x}$

1) $f(x) \geq 0,\ \forall\, x \in [0, 4]$ lorsque $k > 0$

4

2) Il faut que $\int_0^4 k(x+2)e^{-x}\,dx = 1$

Calculons $I = \int (x+2)e^{-x}\,dx$.

$u = x + 2$ $du = dx$	$dv = e^{-x}\,dx$ $v = -e^{-x}$

$I = -(x+2)e^{-x} + \int e^{-x}\,dx$

$I = -(x+2)e^{-x} - e^{-x} + C = -(x+3)e^{-x} + C$

En résolvant $\int_0^4 k(x+2)e^{-x}\,dx = 1$

$$k\left[-(x+3)e^{-x}\right]\Big|_0^4 = 1$$

$$k\left[-7e^{-4} + 3\right] = 1$$

d'où $k = \dfrac{1}{3 - 7e^{-4}}$

b) Soit X, le temps d'attente en minutes.

i) $P(0 \le X \le 2) = \dfrac{1}{3-7e^{-4}}\int_0^2 (x+2)e^{-x}\,dx$

$= \dfrac{1}{3-7e^{-4}}\left[-(x+3)e^{-x}\right]\Big|_0^2$

$= \dfrac{1}{3-7e^{-4}}\left[-5e^{-2} + 3\right]$

$= 0{,}809\,0\ldots$

d'où $P(0 \le X \le 2) \approx 0{,}809$

ii) $P(2 \le X \le 4) = \dfrac{1}{3-7e^{-4}}\left[-(x+3)e^{-x}\right]\Big|_2^4$

$= \dfrac{1}{3-7e^{-4}}\left[-7e^{-4} + 5e^{-2}\right]$

$= 0{,}190\,9\ldots$

d'où $P(2 \le X \le 4) \approx 0{,}191$

Solution 2 : $P(2 \le X \le 4) = 1 - P(0 \le X \le 2)$

$\approx 1 - 0{,}809$

$\approx 0{,}191$

iii) $P(3 < X \le 4) = \dfrac{1}{3-7e^{-4}}\left[-(x+3)e^{-x}\right]\Big|_3^4$

$= \dfrac{1}{3-7e^{-4}}\left[-7e^{-4} + 6e^{-3}\right]$

$= 0{,}059\,3\ldots$

d'où $P(3 < X \le 4) \approx 0{,}059$

c) $\mu = E(X) = \int_0^4 x\,f(x)\,dx = \dfrac{1}{3-7e^{-4}}\int_0^4 x(x+2)e^{-x}\,dx$

Calculons $I = \int (x^2 + 2x)e^{-x}\,dx$.

$u = x^2 + 2x$ $du = (2x+2)\,dx$	$dv = e^{-x}\,dx$ $v = -e^{-x}$

$I = -(x^2+2x)e^{-x} + \int (2x+2)e^{-x}\,dx$

$u = 2x + 2$ $du = 2dx$	$dv = e^{-x}\,dx$ $v = -e^{-x}$

$I = -(x^2+2x)e^{-x} - (2x+2)e^{-x} + \int 2e^{-x}\,dx$

$= -(x^2+2x)e^{-x} - (2x+2)e^{-x} - 2e^{-x} + C$

$= -(x^2+4x+4)e^{-x} + C$

ainsi $\mu = \dfrac{1}{3-7e^{-4}}\left[-(x^2+4x+4)e^{-x}\right]\Big|_0^4$

$= \dfrac{1}{3-7e^{-4}}\left[-36e^{-4} + 4\right]$

$= 1{,}163\,2\ldots$

d'où $\mu \approx 1{,}16$ min

Ce temps correspond au temps d'attente moyen des visiteurs au kiosque d'information touristique.

19. a) $\dfrac{dQ}{dt} = \dfrac{(7-Q)(1-Q)}{1200}$

$$\frac{dQ}{(7-Q)(1-Q)} = \frac{1}{1200}dt$$

$$\int \frac{1}{(7-Q)(1-Q)}dQ = \int \frac{1}{1200}dt$$

Décomposons $\dfrac{1}{(7-Q)(1-Q)}$ en une somme de fractions partielles.

$$\frac{1}{(7-Q)(1-Q)} = \frac{A}{(7-Q)} + \frac{B}{(1-Q)}$$

$$= \frac{\frac{-1}{6}}{(7-Q)} + \frac{\frac{1}{6}}{(1-Q)}$$

Donc $\displaystyle\int\left(\frac{-1}{6(7-Q)} + \frac{1}{6(1-Q)}\right)dQ = \int \frac{1}{1200}dt$

$$\frac{1}{6}\ln|7-Q| - \frac{1}{6}\ln|1-Q| = \frac{1}{1200}t + C$$

Puisque à $t = 0$, $Q = 0$, alors

$\dfrac{1}{6}\ln 7 - \dfrac{1}{6}\ln 1 = 0 + C$, donc $C = \dfrac{1}{6}\ln 7$

Ainsi $\dfrac{1}{6}\ln\left(\dfrac{7-Q}{1-Q}\right) = \dfrac{1}{1200}t + \dfrac{1}{6}\ln 7$ (car $Q < 1$)

$$\ln\left(\frac{7-Q}{1-Q}\right) = \frac{1}{200}t + \ln 7$$

$$\left(\frac{7-Q}{1-Q}\right) = 7e^{\frac{t}{200}}$$

$$7 - Q = 7e^{\frac{t}{200}} - 7Qe^{\frac{t}{200}}$$

$$Q(7e^{\frac{t}{200}} - 1) = 7e^{\frac{t}{200}} - 7$$

d'où $Q = \dfrac{7\left(e^{\frac{t}{200}} - 1\right)}{7e^{\frac{t}{200}} - 1}$ ou $Q = \dfrac{7\left(1 - e^{\frac{-t}{200}}\right)}{7 - e^{\frac{-t}{200}}}$

b) $Q(10) = \dfrac{7\left(1 - e^{\frac{-10}{200}}\right)}{7 - e^{\frac{-10}{200}}} \approx 0{,}0564$

donc environ 5,64 %.

$Q(60) = \dfrac{7\left(1 - e^{\frac{-60}{200}}\right)}{7 - e^{\frac{-60}{200}}} \approx 0{,}289$

donc environ 29 %.

c) Déterminons le temps t_1 tel que $C = 40$ %.

$$\begin{aligned}
0{,}4 &= \frac{7 - 7e^{\frac{-t}{200}}}{7 - e^{\frac{-t}{200}}}\\
2{,}8 - 0{,}4e^{\frac{-t}{200}} &= 7 - 7e^{\frac{-t}{200}}\\
6{,}6e^{\frac{-t}{200}} &= 4{,}2\\
\frac{-t}{200} &= \ln\left(\frac{4{,}2}{6{,}6}\right)\\
t &= -200\ln\left(\frac{4{,}2}{6{,}6}\right)
\end{aligned}$$

donc $t_1 \approx 90$ minutes.
Déterminons le temps t_2 tel que $Q = 60$ %.

$$\begin{aligned}
0{,}6 &= \frac{7 - 7e^{\frac{-t}{200}}}{7 - e^{\frac{-t}{200}}}\\
4{,}2 - 0{,}6e^{\frac{-t}{200}} &= 7 - 7e^{\frac{-t}{200}}\\
6{,}4e^{\frac{-t}{200}} &= 2{,}8\\
\frac{-t}{200} &= \ln\left(\frac{2{,}8}{6{,}4}\right)\\
t &= -200\ln\left(\frac{2{,}8}{6{,}4}\right)
\end{aligned}$$

donc $t_2 \approx 165$ minutes.
Ainsi $t \approx 165 - 90$
d'où environ 75 minutes.

20. a) Soit P la population.

$$\begin{aligned}
\frac{dP}{dt} &= a\left(\frac{P}{2}\right)^2 - 0{,}2P\\
&= \frac{aP^2}{4} - 0{,}2P\\
&= kP^2 - 0{,}2P
\end{aligned}$$

b) $$\frac{dP}{P(kP - 0{,}2)} = dt$$

$$\begin{aligned}
\frac{1}{P(kP - 0{,}2)} &= \frac{A}{P} + \frac{B}{kP - 0{,}2}\\
1 &= A(kP - 0{,}2) + BP
\end{aligned}$$

Si $P = 0$, alors $-0{,}2A = 1$, donc $A = -5$

Si $P = \frac{0{,}2}{k}$, alors $\frac{0{,}2}{k}B = 1$, donc $B = 5k$

$$\begin{aligned}
\int\left(\frac{-5}{P} + \frac{5k}{kP - 0{,}2}\right)dP &= \int dt\\
-5\ln|P| + 5\ln|kP - 0{,}2| &= t + C\\
5\ln\left|\frac{kP - 0{,}2}{P}\right| &= t + C
\end{aligned}$$

En remplaçant t par 0 et P par 5000, nous obtenons

$$5\ln\left|\frac{5000k - 0{,}2}{5000}\right| = C$$

ainsi $5\ln\left|\frac{kP - 0{,}2}{P}\right| = t + 5\ln\left|\frac{5000k - 0{,}2}{5000}\right|$

En remplaçant t par $5\ln 2$ et P par 4000, nous obtenons

$$\begin{aligned}
5\ln\left|\frac{4000k - 0{,}2}{4000}\right| &= 5\ln 2 + 5\ln\left|\frac{5000k - 0{,}2}{5000}\right|\\
\frac{4000k - 0{,}2}{4000} &= \frac{10\,000k - 0{,}4}{5000}\\
20\,000k - 1 &= 40\,000k - 1{,}6\\
k &= 0{,}000\,03
\end{aligned}$$

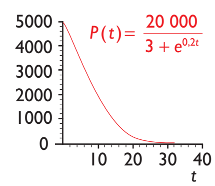
$$\begin{aligned}
5\ln\left|\frac{0{,}000\,03P - 0{,}2}{P}\right| &= t + 5\ln\left|\frac{5000(0{,}000\,03) - 0{,}2}{5000}\right|\\
5\ln\left|\frac{0{,}000\,03P - 0{,}2}{P}\right| &= t + 5\ln|-0{,}000\,01|\\
5\ln\left|\frac{0{,}2 - 0{,}000\,03P}{P}\right| &= t + 5\ln(0{,}000\,01)\\
\ln\left|\frac{0{,}2 - 0{,}000\,03P}{P}\right| &= \frac{t}{5} + \ln(0{,}000\,01)\\
&= \ln\left(e^{\frac{t}{5}}\right) + \ln(0{,}000\,01)\\
&= \ln\left(0{,}000\,01e^{\frac{t}{5}}\right)\\
\frac{0{,}2 - 0{,}000\,03P}{P} &= 0{,}000\,01e^{\frac{t}{5}}\\
0{,}2 &= 0{,}000\,01e^{\frac{t}{5}}P + 0{,}000\,03P\\
0{,}2 &= P(0{,}000\,01e^{\frac{t}{5}} + 0{,}000\,03)\\
P &= \frac{0{,}2}{0{,}000\,01e^{\frac{t}{5}} + 0{,}000\,03}
\end{aligned}$$

d'où $P = \dfrac{20\,000}{3 + e^{0{,}2t}}$

c)
```
> P:=t->20000/(3+exp(0.2*t));
```

$$P := t \to 20\,000\,\frac{1}{3 + e^{(.2t)}}$$

```
> plot(P(t),t=0..40);
```



d) En posant $P = 2500$ dans l'équation de P, nous obtenons

$$\begin{aligned}
2500 &= \frac{20\,000}{3 + e^{\frac{t}{5}}}\\
e^{\frac{t}{5}} &= \frac{20\,000}{2500} - 3\\
\frac{t}{5} &= \ln 5\\
t &= 5\ln 5\\
t &\approx 8{,}05 \text{ ans}
\end{aligned}$$

donc au début de l'an 2016 (2008 + 8).

4