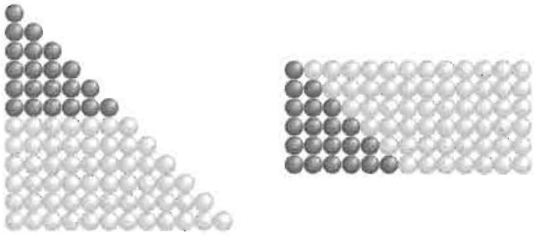


Figure 1.1.

He barely finished stating the problem when young Carl came forward and placed his slate on the teacher's desk, void of calculation, with the correct answer: 5050. When asked to explain, Gauss admitted he recognized the pattern $1 + 100 = 101, 2 + 99 = 101, 3 + 98 = 101$, and so on to $50 + 51 = 101$. Since there are fifty such pairs, the sum must be $50 \cdot 101 = 5050$. The pattern for the sum (adding the largest number to the smallest, the second largest to the second smallest, and so on) is illustrated in Figure 1.1, where the rows of balls represent positive integers.

The number $t_n = 1 + 2 + 3 + \cdots + n$ for a positive integer n is called the nth *triangular number*, from the pattern of the dots on the left in Figure 1.1. Young Carl correctly computed $t_{100} = 5050$. However, this solution works only for n even, so we first prove

Theorem 1.1. *For all* $n \geq 1$, $t_n = n(n+1)/2$.

Proof. We find a pattern that works for any n by arranging two copies of t_n to form a rectangular array of balls in n rows and $n + 1$ columns. Then we have $2t_n = n(n+1)$, or $t_n = n(n+1)/2$. See Figure 1.2. ■

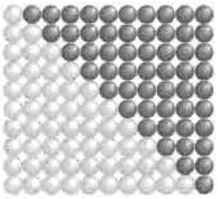
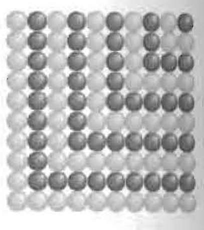


Figure 1.2.

The counting procedure in the preceding combinatorial proof is double counting or the Fubini principle, as mentioned in the Introduction. We employ the same procedure to prove that sums of odd numbers are squares.

Theorem 1.2. *For all* $n \geq 1$, $1 + 3 + 5 + \cdots + (2n-1) = n^2$.

(a)  (b)

Proof. We give two combinatori
in two ways, first as a square arra
in each L-shaped region of simil
one-to one correspondence (illus
triangular array of balls in rows
array of balls. ■

The same idea can be employ
lowing sequence of identities:

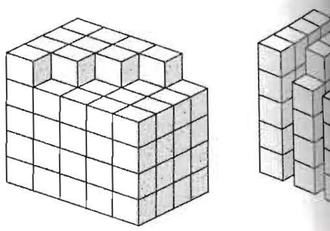
aus

Each row begins w

$n^2 + (n^2 + 1)$

can be proved by i

In Figure 1.4, w
number of small cubes in the pi
$18 + 19 + 20 = 21 + 22 + 23$



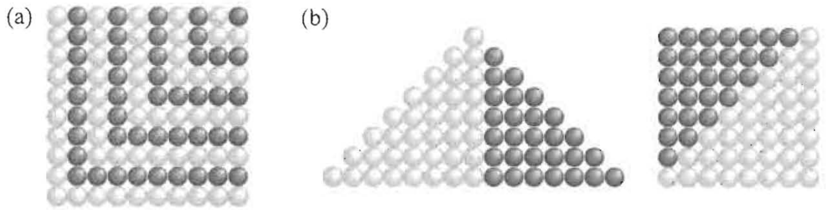


Figure 1.3.

Proof. We give two combinatorial proofs. In Figure 1.3a, we count the balls in two ways, first as a square array of balls, and then by the number of balls in each L-shaped region of similarly colored balls. In Figure 1.3b, we see a one-to one correspondence (illustrated by the color of the balls) between a triangular array of balls in rows with $1, 3, 5, \ldots, 2n-1$ balls, and a square array of balls. ■

The same idea can be employed in three dimensions to establish the following sequence of identities:

$$1+2=3,$$
$$4+5+6=7+8,$$
$$9+10+11+12=13+14+15, \text{ etc.}$$

Each row begins with a square. The general pattern

$$n^2+(n^2+1)+\cdots+(n^2+n)=(n^2+n+1)+\cdots+(n^2+2n)$$

can be proved by induction, but the following visual proof is nicer.

In Figure 1.4, we see the $n=4$ version of the identity where counting the number of small cubes in the pile in two different ways yields $16+17+18+19+20=21+22+23+24$.

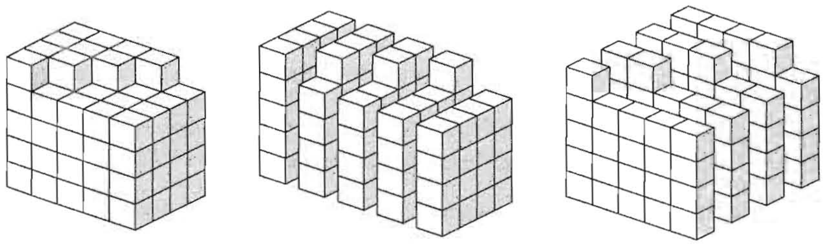


Figure 1.4.

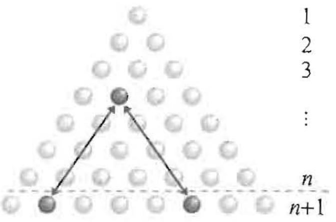


Figure 1.8.

The arrows denote the correspondence between an element of the set with t_n elements and a pair of elements from a set of $n + 1$ elements. ■

1.2 Sums of squares, triangular numbers, and cubes

Having examined triangular numbers and squares as sums of integers and sums of odd integers, we now consider sums of triangular numbers and sums of squares.

Theorem 1.7. *For all* $n \geq 1$, $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

Proof. We give two proofs. In the first we exhibit a one-to-one correspondence between three copies of $1^2 + 2^2 + 3^2 + \cdots + n^2$ and a rectangle whose dimensions are $2n + 1$ and $1 + 2 + \cdots + n = n(n+1)/2$ [Gardner, 1973]. See Figure 1.9.

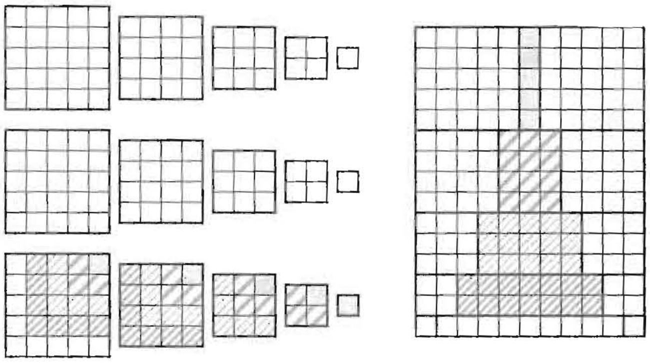


Figure 1.9.

Hence $3(1^2 + 2^2 + 3^2 + \cdots + n^2) = ($
the result follows.

In the second proof, we write each sq
those numbers in a triangular array, cre
triangular array by 120° and 240°, and
triangular array. See Figure 1.10 [Kung,

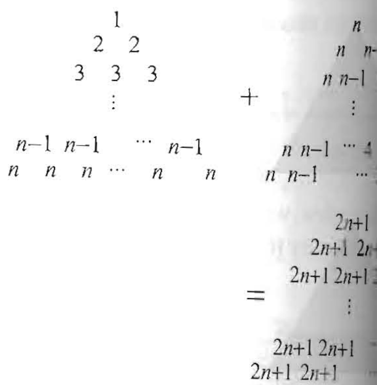


Figure 1.

Theorem 1.8. *For all* $n \geq 1$, $t_1 + t_2 + t_3$

Proof. In Figure 1.11, we stack layers o
gular numbers. The sum of the triangular
which is the same as the total volume of t
we "slice off" small pyramids (shaded gr
on the top of the cube from which it came
gular pyramid minus some smaller right tr
of the base.

Thus $t_1 + t_2 + \cdots + t_n = \frac{1}{6}(n+1)^3$

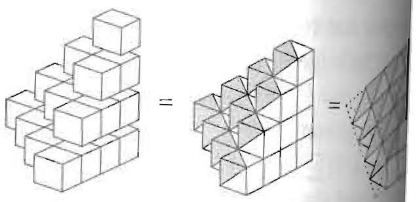


Figure 1.1

Hence $3(1^2 + 2^2 + 3^2 + \cdots + n^2) = (2n + 1)(1 + 2 + \cdots + n)$ from which the result follows.

In the second proof, we write each square k^2 as a sum of k ks, then place those numbers in a triangular array, create two more arrays by rotating the triangular array by 120° and 240°, and add corresponding entries in each triangular array. See Figure 1.10 [Kung, 1989]. ■

$$\begin{array}{c} 1 \\ 2\quad 2 \\ 3\quad 3\quad 3 \\ \vdots \\ n{-}1\quad n{-}1\quad \cdots\quad n{-}1 \\ n\quad n\quad n\quad \cdots\quad n\quad n \end{array} + \begin{array}{c} n \\ n\quad n{-}1 \\ n\quad n{-}1\quad n{-}2 \\ \vdots \\ n\quad n{-}1\quad \cdots\quad 4\quad 3\quad 2 \\ n\quad n{-}1\quad \cdots\quad 3\quad 2\quad 1 \end{array} + \begin{array}{c} n \\ n{-}1\quad n \\ n{-}2\quad n{-}1\quad n \\ \vdots \\ 2\quad 3\quad 4\quad \cdots\quad n{-}1\quad n \\ 1\quad 2\quad 3\quad \cdots\quad n{-}1\quad n \end{array}$$

$$= \begin{array}{c} 2n{+}1 \\ 2n{+}1\quad 2n{+}1 \\ 2n{+}1\quad 2n{+}1\quad 2n{+}1 \\ \vdots \\ 2n{+}1\quad 2n{+}1\quad \cdots\quad 2n{+}1 \\ 2n{+}1\quad 2n{+}1\quad \cdots\quad 2n{+}1 \end{array}$$

Figure 1.10.

Theorem 1.8. *For all $n \geq 1$, $t_1 + t_2 + t_3 + \cdots + t_n = \dfrac{n(n+1)(n+2)}{6}$.*

Proof. In Figure 1.11, we stack layers of unit cubes to represent the triangular numbers. The sum of the triangular numbers is total number of cubes, which is the same as the total volume of the cubes. To compute the volume, we "slice off" small pyramids (shaded gray) and place each small pyramid on the top of the cube from which it came. The result is a large right triangular pyramid minus some smaller right triangular pyramids along one edge of the base.

Thus $t_1 + t_2 + \cdots + t_n = \dfrac{1}{6}(n+1)^3 - (n+1)\dfrac{1}{6} = \dfrac{n(n+1)(n+2)}{6}$. ■

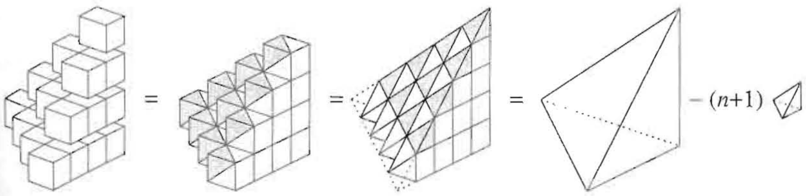


Figure 1.11.

In the proof we evaluated the sum of the first n triangular numbers by computing volumes of pyramids. This is actually an extension of the Fubini principle from simple enumeration of objects to additive measures such as length, area and volume. The volume version of the Fubini principle is: *computing the volume of an object in two different ways yields the same number*; and similarly for length and area. We cannot, however, extend the Cantor principle to additive measures—for example, one can construct a one-to-one correspondence between the points on two line segments with different lengths.

Theorem 1.9. *For all* $n \geq 1$, $1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2 = t_n^2$.

Proof. Again, we give two proofs. In the first, we represent k^3 as k copies of a square with area k^2 to establish the identity [Cupillari, 1989; Lushbaugh, 1965].

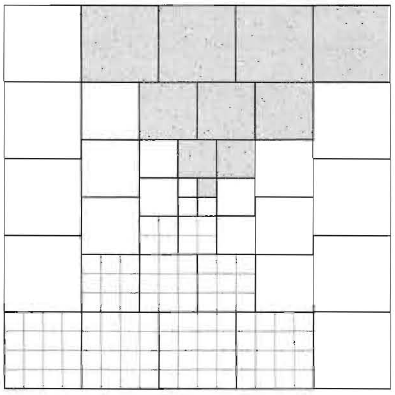


Figure 1.12.

In Figure 1.12, we have $4(1^3 + 2^3 + 3^3 + \cdots + n^3) = [n(n+1)]^2$ (for $n = 4$).

For the second proof, we use the fact that $1 + 2 + 3 + \cdots + (n-1) + n + (n-1) + \cdots + 2 + 1 = n^2$ (see Challenge 1.1a) and consider a square array of numbers in which the element in row i and column j is ij, and sum the numbers in two different ways [Pouryoussefi, 1989].

Summing by columns yields $\sum_{i=1}^n i + 2(\sum_{i=1}^n i) + \cdots + n(\sum_{i=1}^n i) = (\sum_{i=1}^n i)^2$, while summing by the L-shaped shaded regions yields (using the result of Challenge 1.1a) $1 \cdot 1^2 + 2 \cdot 2^2 + \cdots + n \cdot n^2 = \sum_{i=1}^n i^3$. ■

$$\begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & \vdots & \vdots & & \vdots \\ n & 2n & 3n & \cdots & n^2 \end{array}$$

Figure 1.1

We conclude this section with a theore
sum of integers.

Theorem 1.10. *For all* $n \geq 1$, $\sum_{i=1}^n \sum$

Proof. We represent the double sum as a
pute the volume of a rectangular box com
tion. See Figure 1.14.

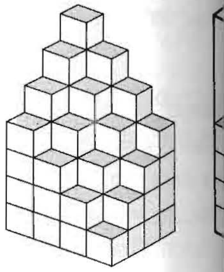


Figure 1.

Two copies of the sum $S = \sum_{i=1}^n \sum$
box with base n^2 and height $2n$, hence c
two ways yields $2S = 2n^3$, or $S = n^3$.

1.3 There are infinitely m

Reductio ad absurdum
a mathematician's fi
than any chess gambi
of a pawn or even a
game.

The earliest proof that there are infinitel
in the *Elements* (Book IX, Proposition